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MODULE OF HOMEOMORPHISMS TO MODULE

Abstract

In this article, after a concise presentation of the modules over rings as a generalization of vector space over the fields, their homeomorphisms are treated. Further builds R-module \( R \)-module of morphisms of the modules.

Keywords: R-module, left (right) R-module, abelian group, associative ring, R-homeomorphisms

1. The meaning of the R-Module, feature

Let \( M \) be an non empty set of equipped with an internal algebraic action \([2]\) marked with the symbol of collection \( + \) and \( R \) an associative ring \([3]\). A set \( M \) is also equipped with an algebraic external action \([2]\) indicated by the multiplication symbol \( \cdot \), which, when reflecting \( R \times M \) in \( M \), is referred to as the left multiplication in \( M \) with elements from \( R \), whereas, when reflecting the \( M \times R \) in \( M \) is called right multiplication in \( M \) with elements from \( R \). In the first case the couple’s image \((r, m) \in R \times M \) is written \( r \cdot m \), in the second case the couple’s image \((m, r) \in M \times R \) is written \( m \cdot r \).

**Definition 1.1** \([1, 5, 6]\) In the above conditions, the left module above the \( R \) ring is called the structure \((M, +, \cdot)\), which has its own attributes:

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• \((M, +)\) is an abelian group; \hfill (1)
• \(\forall (r_1, r_2, m) \in R^2 \times M, \, r_1(r_2m) = (r_1r_2)m\); \hfill (2)
• \(\forall (r, m_1, m_2) \in R \times M^2, \, r(m_1 + m_2) = rm_1 + rm_2\); \hfill (3)
• \(\forall (r_1, r_2, m) \in R^2 \times M, \, (r_1 + r_2)m = r_1m + r_2m\). \hfill (4)

**Definition 1.2.** Under the above conditions, the right module above the ring is called the structure \((M, +, \cdot)\), which has its own attributes:

• \((M, +)\) is an abelian group; \hfill (1’)
• \(\forall (m, r_1, r_2) \in M \times R^2, \, (mr_1)r_2 = m(r_1r_2)\); \hfill (2’)
• \(\forall (m_1, m_2, r) \in M^2 \times R, \, (m_1 + m_2)r = m_1r + m_2r\); \hfill (3’)
• \(\forall (m, r_1, r_2) \in M \times R^2, \, m(r_1 + r_2) = mr_1 + mr_2\). \hfill (4’)

The left (right) module above the ring is marked \(R M \ (M_R)\) and is called \(R\)-left module (right). If the left-hand module above \(R\) is also the right is called a module above the ring, in short \(R\)-module.

If the ring has a single element \(1_R\) (short \(1\)) and the above-mentioned attributes for \(R M \ (M_R)\) is added the feature

• \(\forall m \in M, \, 1 \cdot m = m (\ m \cdot 1 = m)\) \hfill (5)

then the module \(R M \ (M_R)\) is called the unitary left (right) module above the ring.

In ongoing, the \(R\) ring is associated and for a module on such a ring simple naming is used \(R\)-Module.

Below we will treat the \(R\)-modules, implying left \(R\)-modules, since the right \(R\)-modules are treated analogously.

**THEOREM 1.1.** A \(R\)-module \(M\) enjoys the following attributes:

• \(\forall m \in M, \, 0_R \cdot m = 0_M\); \hfill (6)
• \(\forall r \in R, \, r \cdot 0_M = 0_M\); \hfill (7)
• \(\forall m \in M, \forall r \in R, \, (-r) \cdot m = r \cdot (-m) = -r \cdot m \in M\). \hfill (8)
Proof. Let $r$ be a fixed element of the $R$ ring and $m$ any other element of the $R$ module. By Definition 1.1. we have $r \cdot m + 0_R \cdot m = (r + 0_R) \cdot m = r \cdot m$. On the other hand, by the additive group $(M, +)$, we have $r \cdot m + 0_M = r \cdot m$. From here $r \cdot m + 0_R \cdot m = r \cdot m + 0_M$, that gives $0_R \cdot m = 0_M$:

- $r \cdot 0_M = r \cdot (0_R \cdot m) = (r \cdot 0_R) \cdot m = 0_R \cdot m = 0_M$.
- $r \cdot m + (-r) \cdot m = r + (-r)m = 0_R \cdot m = 0_R \cdot m = 0_M$

$\Rightarrow (-r) \cdot m = -r \cdot m$.

2. $R$-Homeomorphisms of $R$-Modules

Definition 2.1 [1,6] $R$-homomorphism (or $R$-morphism) of a $R$-module $M$ in a $R$-module $N$ is called any reflection $f: M \rightarrow N$ having attributes

- $\forall m_1, m_2 \in M;\quad (9)$
- $f(r \cdot m) = r \cdot f(m), \forall r \in R$ and $\forall m \in M$ (10)

(ose $f(m \cdot r) = f(m) \cdot r, \forall r \in R$ and $\forall m \in M$).

If $M=N$, then the reflection $f$ is called $R$-endomorphism in $M$.

THEOREM 2.1. For every two $R$-modules $M, N$, if the reflection $f: M \rightarrow N$ is a $R$-homomorphism, then

- $f(0_M) = 0_N$,
- $f(-m) = -f(m), \forall m \in M$,
- $f(m_1 - m_2) = f(m_1) - f(m_2), \forall m_1, m_2 \in M$.

Proof. According to (6) and (10) we have

$f(0_M) = f(0_R \cdot m) = 0_N, f(0_M) = 0_N$.

Further, according to (9),

$0_N = f(0_M) = \bar{u}\bar{u}\bar{u} + (-) = (+) + (-)$,

that tells us $f(-m)$ is the symmetric of $f(m)$ in the group $(N, +)$, so $-f(m) = f(-m)$. Finally,

$f(m_1 - m_2) = f(m_1) - f(m_2), \forall m_1, m_2 \in M$.

THEOREM 2.2. For each two $R$-modules $M, N$, reflection $p_0: M \rightarrow N$, defined by $p_0(m) = 0_N, \forall m \in M$, is the $R$-homeomorphism of $M$ to $N$.

Proof. From the above definition of reflection $p_0$ we have
\[ p_0(m_1 + m_2) = 0_N = 0_N + 0_N = p_0(m_1) + p_0(m_2), \quad \forall m_1, m_2 \in M, \]

which indicates that \( p_0 \) enjoys the attribute (9); we also have

\[ p_0(r \cdot m) = 0_N = r \cdot 0_N = r \cdot p_0(m), \quad \forall r \in R \text{ dhe } \forall m \in M, \]

which indicates that \( p_0 \) also enjoys the attribute (10).

**THEOREM 2.3.** Identical reflection \( I_M : M \to M \) (e.g. the reflection defined by \( I_M(m) = m, \forall m \in M \)) is an \( R \)-endomorphism in \( M \).

**Proof.** From the above definition of the identical reflection \( I_M \) we have

\[ I_M(m_1 + m_2) = m_1 + m_2 = I_M(m_1) + I_M(m_2), \quad \forall m_1, m_2 \in M, \]

Indicating that the \( I_M \) enjoys the attribute (9); we also have

\[ I_M(r \cdot m) = r \cdot m = r \cdot I_M(m), \quad \forall r \in R \text{ dhe } \forall m \in M, \]

which indicates that \( I_M \) enjoys the attribute (10).

### 3. Module \( \text{Hom}_R(M, N) \) of \( R \)-Homeomorphisms of the Modules

The study of homomorphisms of modules bring to the construction of an important module, called the **homomorphism module**.

Let be given the \( R \)-module \( M \) and the \( R \)-module \( N \). The set of \( R \)-homomorphisms from \( M \) to \( N \) is written \( \text{Hom}_R(M, N) \).

**Definition 3.1.** Let be \( f, g \) two possible reflections from \( M \) to \( N \) and \( r \) an element of an \( R \) ring. Then:

1. Many of the reflection \( f \) with the \( g \) reflection, which is written \( f + g \), is called reflection \( f+g : M \to N \), defined by
   \[ (f + g)(m) = f(m) + g(m), \quad \forall m \in M. \tag{14} \]
2. The opposite reflection of \( f \) reflection, which is written \(-f\) is called reflection \(-f : M \to N \), defined by
   \[ (-f)(m) = -f(m), \quad \forall m \in M. \tag{15} \]
3. The left product of the reflection \( f \) with the element \( r \in R \), which is written \( rf \), is called the reflection \( rf : M \to N \), defined by
   \[ (rf)(m) = rf(m), \quad \forall m \in M. \tag{16} \]

An analogy is given to the meaning and the right production \( fr \) such that

\[ (fr)(m) = f(m) \cdot r, \quad \forall m \in M. \]

**THEOREM 3.1.** If the reflections \( f, g \) are \( R \)-homomorphisms from \( M \) to \( N \) then:

1. \( f+g \in \text{Hom}_R(M, N) \), \hspace{1cm} \( (17) \)
   otherwise, their amount \( f+g \) is a \( R \)-homomorphism from \( M \) to \( N \);
2. \( -f \in \text{Hom}_R(M, N) \), \hspace{1cm} \( (18) \)
   otherwise, the reverse reflection \(-f \) is a \( R \)-homomorphism from \( M \) to \( N \);
3. For each $r \in R$, where $R$ is commutative,

$$rf \in Hom_R(M, N),$$

(19)

otherwise, the left (right) production of $f$ reflection with elements from $R$ is a $R$-homomorphism from $M$ to $N$.

**Proof.**

1. Since the reflections $f, g$ are $R$-homomorphisms from $M$ to $N$, then

$$\begin{align*}
(f + g)(m_1 + m_2) & \overset{(14)}{=} f(m_1 + m_2) + g(m_1 + m_2) \\
& \overset{(9)}{=} [f(m_1) + f(m_2)] + [g(m_1) + g(m_2)] \\
& \overset{(14)}{=} [f(m_1) + g(m_1)] + [f(m_2) + g(m_2)] \\
& = (f + g)(m_1) + (f + g)(m_2), \ \forall m_1, m_2 \in M,
\end{align*}$$

which shows that $f+g$ enjoys the attribute (9); we also have

$$\begin{align*}
(f + g)(rm) & \overset{(14)}{=} f(rm) + g(rm) \\
& \overset{(10)}{=} rf(m) + rg(m) \\
& = rf(m) + rg(m) \\
& = r[f(m) + g(m)] \\
& = r[(f + g)(m)], \ \forall r \in R \text{ dhe } \forall m \in M,
\end{align*}$$

which shows that $f+g$ enjoys the attribute (10). Consequently $f+g \in Hom_R(M, N)$

2. Reflection $f$ is $R$-homomorphism from $M$ to $N$, therefore

$$\begin{align*}
(-f)(m_1 + m_2) & \overset{(15)}{=} -f(m_1 + m_2) = (-f)(m_1) + (-f)(m_2) \\
& \overset{(9)}{=} f(-m_1) + f(-m_2) = (-f)(m_1) + (-f)(m_2), \ \forall m_1, m_2 \in M,
\end{align*}$$

which shows that $-f$ enjoys the attribute (9); Also, having in mind and (8) we have

$$\begin{align*}
(-f)(r^* \cdot m) & \overset{(12)}{=} f(-r^* \cdot m) = f(r^* \cdot (-m)) = r^* \cdot f(-m) = r^* \cdot [-f(m)] \\
& \overset{(15)}{=} r^* \cdot [-f(m)], \ \forall r \in R \text{ dhe } \forall m \in M,
\end{align*}$$

which shows that $-f$ enjoys even the attribute (10).

3. We also have

$$\begin{align*}
(rf)(m_1 + m_2) & \overset{(16)}{=} rf(m_1 + m_2) = r[f(m_1) + f(m_2)] = rf(m_1) + rf(m_2) \\
& \overset{(9)}{=} (rf)(m_1) + (rf)(m_2), \ \forall m_1, m_2 \in M.
\end{align*}$$
showing that $r\cdot f$ has its attribute (9); also, knowing that the $R$ ring is commutative we have

$$\begin{align*}
(r\cdot f)(\rho m) &= r\cdot [\rho f(m)] \\
&= (r\rho)f(m) = (\rho r)f(m) \\
&= \rho \cdot [r\cdot f(m)], \quad \forall \rho \in R \text{ and } \forall m \in M,
\end{align*}$$

which shows that $r\cdot f$ also enjoys attribute (10).

**Definition 3.2.** $R$-homomorphism $f+g: M \to N$ is called $R$-homeomorphism $f: M \to N$ with $R$-homeomorphism $g: M \to N$, $R$-homeomorphism $r\cdot f$ ( $f\cdot r$ ), when $R$ is commutative, is called left (right) production of $R$-homomorphism $f: M \to N$ with element $r \in R$.

Through this definition, they are introduced into the set $Hom_R(M, N)$ action of addition $+$ and left (right) multiplication, which make it algebra $(, +, \cdot)$ with two actions.

**Theorem 3.2.** If the $R$ ring is commutative, then the algebra $(, +, \cdot)$ of $R$-homeomorphisms from $M$ to $N$ is the $R$-left(right) module.

**Proof.** We show that they satisfy the conditions (1), (2), (3), (4) of Definition 1.1. of a left $R$-module.

(1) From the above it is easy to see that:
- $\forall f, g, h \in Hom_R(M, N), (f + g) + h = f + (g + h)$;
- $\forall f \in Hom_R(M, N), f + p_0 = f$;
- $\forall f \in Hom_R(M, M), f + (-f) = p_0$;
- $\forall f, g \in Hom_R(M, M), f + g = g + f$,
indicating that $Hom_R(M, N), +$ is an abelian group.

(2) $\forall (r_1, r_2, f) \in R^2 \times Hom_R(M, N)$, writing $g = r_2 \cdot f$ , we have

$$\begin{align*}
[r_1 \cdot (r_2 \cdot f)](m) &= (r_1 \cdot g)(m) = r_1 \cdot g(m) = r_1 \cdot [(r_2 \cdot f)(m)] \\
&= r_1 \cdot [f(r_2 \cdot m)] = f(r_1 \cdot r_2 \cdot m) = (r_1 \cdot r_2) \cdot f(m) \\
&= [(r_1 \cdot r_2) \cdot f](m), \quad \forall m \in M,
\end{align*}$$

which indicates that $r_1 \cdot (r_2 \cdot f) = (r_1 \cdot r_2) \cdot f$.

(3) $\forall (r, f, g) \in R \times [Hom_R(M, N)]^2$ we have

$$[r \cdot (f + g)](m) = r \cdot [(f + g)(m)] = r \cdot [f(m) + g(m)] = r \cdot f(m) + r \cdot g(m)$$
\begin{equation}
(16)
(r \cdot f)(m) + (r \cdot g)(m) = (r \cdot f + r \cdot g)(m), \quad \forall m \in M,
\end{equation}

which indicates that \( r \cdot (f+g) = r \cdot f + r \cdot g \).

\begin{equation}
(4)
\forall (r_1, r_2, f) \in R^2 \times \text{Hom}_R(M, N), \quad \text{we have}
\end{equation}

\begin{align*}
[(r_1 + r_2) \cdot f](m) & = (r_1 + r_2) \cdot f(m) = f((r_1 + r_2)m) = f(r_1 m + r_2 m) = f(r_1 m) + f(r_2 m) \\
& = r_1 \cdot f(m) + r_2 \cdot f(m) = (r_1 \cdot f)(m) + (r_2 \cdot f)(m) = (r_1 \cdot f + r_2 \cdot f)(m), \quad \forall m \in M,
\end{align*}

which indicates that \( (r_1 + r_2) \cdot f = r_1 \cdot f + r_2 \cdot f \).

Analogously it is shown that \( (\text{Hom}_R(M, N), +, \cdot) \) is the right \( R \)-module
when \( \cdot \) is right multiplication with elements from \( R \).
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