New Paths Opened by $\overline{g} - Function$ in Pseudo-Analysis and Other Fields Through Several Applications

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Abstract

The extension problem for the axiomatic concepts of pseudo-arithmetic operations $\{\overline{\oplus}, \overline{\odot}, \overline{\ominus}, \overline{\oslash}\}$ supported by g⁻-Functions are treated in the field of Pseudo-Analysis by many authors, opening new paths for development and investigation of their role as well as for the modifying and modified functions. The g⁻-Negation of the negation N is presented in this paper transformed by the g⁻-Function as a general or normed generator g⁻=g⁻(a,r) in Pseudo-Analysis, where the role of the extended pseudo-arithmetic operations sistem $\{\oplus\_g\ ̄, \odot\_g\ ̄, \ominus\_g\ ̄ (,\oslash)\_g\ ̄\}$ is very specific and important for development of g⁻-Calculus. Furthermore, developing the theory of action of these special functions (g⁻,f_g⁻,t_g⁻,N_g⁻) by generalizations and modifications, we arrive at some connections of Generated Pseudo-Analysis with other fields such as Information Theory, Geometry, Trigonometry, Elementary Algebra and other areas of pure mathematics connected with combinatorial problems. In these fields, the paper addresses several composition of some real continuous parameterized functions with other special functions, above all showing the interesting forms of their generalization and transformation created by modification through g⁻-Transform. Some important formula and classical problems are generalized and transformed, leading us to new connections between different problems and fields.

Keywords: *Pseudo-Analysis, pseudo-operations, transform, g⁻-Function, g⁻-Negation*

1. Introduction

Pseudo-Analysis, as a generalization of the Classical Analysis has very important results in its two branches, Generated and Idempotent Pseudo-Analysis where the role of the consistent system of pseudo-arithmetic operations (generated by the generator g or idempotent) has brought interestings developments in the Theory of Pseudo-Additive Measures and Integrals [5], [22]. The concept of pseudo-arithmetical operations $\{\oplus, \odot, \ominus, \oslash\}$ as a sistem generated by the generator

g first was introduced on $[0,+\infty]$ interval and then to the whole extended real line $\bar{R} = [-\infty, +\infty]$ [2], [3], [4], [7], [8], [11], [13], [15], [17] and the generator $\bar{g}$ is extended in $\bar{R}$. The generator g of the binary operation $\oplus$, was extended into the odd function $\bar{g}$ such that: $\bar{g}(x) = \begin{cases} g(x), & \text{for } x \in [0,\infty] \\ -g(-x), & \text{for } x \in [-\infty, 0) \end{cases}$, or briefly $\bar{g}(x) = sgn\, x \cdot g(|x|),\ x \in [-\infty, \infty]$ [11], [21], [27].

The role of the pseudo-arithmetic operations $\{\overline{\oplus}, \overline{\odot}, \overline{\ominus}, \overline{\oslash}\} = \{\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}}, \overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\}$ is shown directly by taking the Rybárik [4] rational functions and Pap [6], but $\bar{g} - Transform$ is a further development of $g - Calculus$. Pseudo-arithmetical operations are useful tools in treating [2] of nonlinear problems and some elementary $\bar{g} - functions$ are derived as solutions of some functional equations using results of Aczél [1]. So, the binary operation $\overline{\oplus}$ defined on the interval $[-\infty, \infty]$ by: $x \overline{\oplus} y = x \overline{\oplus}_{\bar{g}} y = \bar{g}^{-1}[\bar{g}(x) + \bar{g}(y)]$ is also commutative, associative and continuous. The extended forms of $\bar{g} - calculus$ [3], [4], [6], [7], [14], [17], [19], [21]:
$x \overline{\oplus}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x) + \bar{g}(y));\ x \overline{\odot}_{\bar{g}} y = \bar{g}^{-1}(g(x) \cdot \bar{g}(y));$
$x \overline{\ominus}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x) - \bar{g}(y)); x \overline{\oslash}_{\bar{g}} y = \bar{g}^{-1}(\bar{g}(x)/\bar{g}(y)).$

The presentation of $\bar{g} - function$ $(f_{\bar{g}})$ corresponding to the function f as modified functions by $\bar{g} - transform$, lead the $\overline{\bar{g}} - calculus$ to express to another form by using $\bar{g} - Transform$ for $\boldsymbol{t} \circ f$ [10]. For further investigation, we use the generator (normed generator) on $[-\infty, +\infty]$.

2. Materials and Methods

The study method consists in the systematization of the theoretical material, the bringing of some important concepts, functions, combination of them and knowledge from Pseudo-Analysis, specifically Generated Pseudo-Analysis and finding connections with other fields. This connection has been highlighted by us as a result of finding some quite interesting applications of the pseudo-analysis apparatus in other fields, like Pseudo-Linear Algebra, Information Theory, Geometry, Trigonometry, Elementary Algebra and other areas of pure mathematics connected with combinatorial problems. All implementations are listed according to the fields of application and important relationships are identified in detail for each case and according to the nature of the $\bar{g} - generator$.

2.1 First off all let us introduce some definitions

Let f be a function on $]a, b[\subseteq]-\infty, +\infty[$ and the function $\bar{g}$ be a generator of the consistent system of pseudo-arithmetical operations $\{\overline{\oplus}_{\bar{g}}, \overline{\odot}_{\bar{g}}, \overline{\ominus}_{\bar{g}}, \overline{\oslash}_{\bar{g}}\}$.

The function $f_{\bar{g}}$ given by $f_{\bar{g}}(x) = \bar{g}^{-1}\left(f(\bar{g}(x))\right)$ for every $x \in \left(\bar{g}^{-1}(a), \bar{g}^{-1}(b)\right)$ is said to be $\overline{\boldsymbol{g}} - \boldsymbol{function}$ corresponding to the function f.

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In this paper are treated the real functions which are continuous from $\overline{R}$ to $\overline{R}$, or functions from $\overline{R}$ to $\overline{R} \times \overline{R}$.

Let f and k be two continuous functions $f: \overline{R} \to \overline{R}$, $k: \overline{R} \times \overline{R} \to \overline{R}$ and let $\bar{g} - generator$ be extended on $\overline{R}$ (perhaps with some undefined values). The $\bar{g} - \boldsymbol{composite}\ function\ h_{\bar{g}}$ from $\overline{R}$ to $\overline{R} \times \overline{R}$, for $h(x, y) = (f \circ k)(x, y)$ is a function satisfying: $h_{\bar{g}}(x, y) = \left((f \circ k)(x, y)\right)_{\bar{g}} = \left(f_{\bar{g}} \circ k_{\bar{g}}\right)(x, y)$. The composition of functions is not commutative, but associative.

The parameterized nonlinear continuous functions $\boldsymbol{t}$, $\boldsymbol{t}: \overline{R} \times \overline{R} \to \overline{R}$, $a_1, a_2, a_3, a_4 \in R$ are defined in the form bellow:
$\boldsymbol{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(x, y) = a_1 \cdot x \cdot y + a_2 \cdot x + a_3 \cdot y + a_4.$
By the definition for the composition of the two our functions $\boldsymbol{t}$ and f we get [10]:
$h(x, y) = (\boldsymbol{t} \circ f)(x, y) = \boldsymbol{t}(f(x), f(y)) = \boldsymbol{t}^{(+, \cdot)(a_1, a_2, a_3, a_4)}(f(x), f(y)).$
Also, by the definition of the function $\boldsymbol{t}$ we have the form:
$t^{(+, \cdot)(a_1, a_2, a_3, a_4)}(f(x), f(y)) = a_1 \cdot f(x) \cdot f(y) + a_2 \cdot f(x) + a_3 \cdot f(y) + a_4.$

When $\bar{g} - transform$ is applied for $h = t \circ f$, the definition of $\bar{g} - function$ brings us the equation of $h_{\bar{g}}$ as:

$$h_{\bar{g}}(x,y) = \left(t^{(+,\cdot)(a_1,a_2,a_3,a_4)}(f(x),f(y)) \right)_{\bar{g}} =$$

$$= t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})\left(\bar{g}^{-1}(a_1),\ \bar{g}^{-1}(a_2),\bar{g}^{-1}(a_3),\bar{g}^{-1}(a_4) \right)}(f_{\bar{g}}(x),f_{\bar{g}}(y)).$$

For the special cases when $f(x) = \bar{g}(x); f(y) = \bar{g}(y)$, we have:

$\bar{g}^{-1}(\bar{g}(x)) = \bar{g}_{\bar{g}}(x) = x_{\bar{g}} = x$, and $\bar{g}^{-1}(\bar{g}(y)) = \bar{g}_{\bar{g}}(y) = y_{\bar{g}} = y$.

So, $t_{\bar{g}}$ can be written:

$$t^{(+,\cdot)(a_1,a_2,a_3,a_4)}(\bar{g}(x),\bar{g}(y)) = a_1 \cdot \bar{g}(x) \cdot \bar{g}(y) + a_2 \cdot \bar{g}(x) + a_3 \cdot \bar{g}(y) + a_4$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})\left(\bar{g}^{-1}(a_1),g^{-1}(a_2),\bar{g}^{-1}(a_3),\bar{g}^{-1}(a_4) \right)}(x,y) = \qquad =$$

$$\left(\bar{g}^{-1}(a_1) \ \overline{\odot}_{\bar{g}} \ x \ \overline{\odot}_{\bar{g}} \ y \right) \oplus_{\bar{g}} \left(\bar{g}^{-1}(a_2) \ \overline{\odot}_{\bar{g}} \ x \right) \oplus_{\bar{g}} \left(\bar{g}^{-1}(a_3) \ \overline{\odot}_{\bar{g}} \ y \right) \ \oplus_{\bar{g}} \ \left(\ \bar{g}^{-1}(a_4) \right).$$

For $\overline{g} - normed$, the form of t and $t_{\bar{g}}$ is:

$$t^{(+,\cdot)(1,1,1,1)}(x,y,z) = x \cdot y \cdot z + x + y + z + 1$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(1,1,1,1)}(x,y) = \left(x \ \overline{\odot}_{\bar{g}} \ y \ \overline{\odot}_{\bar{g}} \ z \right) \oplus_{\bar{g}} (x) \ \oplus_{\bar{g}} (y) \ \oplus_{\bar{g}} (z) \ \oplus_{\bar{g}} (1).$$
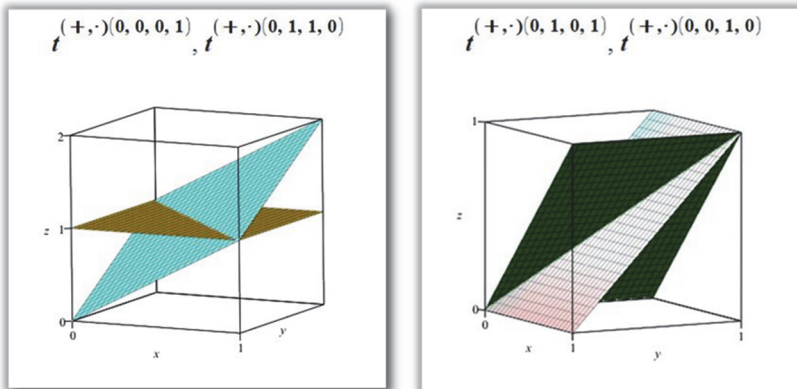

Figure 1.2. Same parameterized nonlinear continuous functions **t**

2.1.1   Conection of $\overline{g} - calculus$ and Parameterized Nonlinear Functions

Based on the extended forms of $\overline{g} - calculus$ and the definition of the Parameterized Nonlinear Functions can be presented the relations between them. For the most important cases of identity, i.e., when $f(x) = \bar{g}(x) = x, f(y) = \bar{g}(y) = y$, we take the functional equation $t(f(x),f(y)) = t(x,y)$ [10] and easily, can verify that $\overline{g} - calculus$ by using $\bar{g} - Transform$ for $t \circ f$, lead to the follow forms:

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,1,1,0)}(x,y) = \bar{g}^{-1}\left(t^{(+,\cdot)(0,1,1,0)}(\bar{g}(x),\bar{g}(y)) \right) =$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,1,0,0)}(x,y) \ \overline{\oplus}_{\bar{g}} \ t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,0,1,0)}(x,y);$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(1,0,0,0)}(x,y) = \bar{g}^{-1}\left(t^{(+,\cdot)(1,0,0,0)}(\bar{g}(x),\bar{g}(y)) \right) =$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,1,0,0)}(x,y) \ \overline{\odot}_{\bar{g}} \ t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,0,1,0)}(x,y);$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,1,-1,0)}(x,y) = \bar{g}^{-1}\left(t^{(+,\cdot)(0,1,1,0)}(\bar{g}(x),\bar{g}(y)) \right) =$$

$$t_{g}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,1,0,0)}(x,y) \ \overline{\ominus}_{\bar{g}} \ t_{\bar{g}}^{(\oplus_{\bar{g}},\overline{\odot}_{\bar{g}})(0,0,1,0)}(x,y);$$

$$t_{\bar{g}}^{(\oplus_{\bar{g}}, \bigcirc_{\bar{g}})(0,1,1,0)}(x, 1/y) = \bar{g}^{-1}\left(t^{(+,\cdot)(0,1,1,0)}\left(\bar{g}(x), 1/\bar{g}(y)\right)\right) = t_{\bar{g}}^{(\oplus_{\bar{g}}, \bigcirc_{\bar{g}})(0,1,0,0)}(x, 1/$$

$$y)\, \overline{\bigcirc}_{\bar{g}}\, t_{\bar{g}}^{(\oplus_{\bar{g}}, \bigcirc_{\bar{g}})(0,0,1,0)}(x, 1/y).$$

2.2 Some Connections of Pseudo-Analysis with other Fields by Application of Generalizations and Modifications from $\bar{g} - Transform$

2.2.1   Application in Pseudo-Linear Algebra. The $\bar{g} - Transform$ of the Linear Sistems

Definition 2.2.1.1. [27] A function $\bar{g}$ is a generator on $[-\infty, +\infty]$ if and only if it is odd and strictly increasing bijection on interval $(-\infty, +\infty)$.

Remark 2.2.1.2. [27] If $\bar{g}, \bar{q}$ are generators, then $\bar{g}^{-1}$ and $\bar{g} \circ \bar{q}^{-1}$ are also generators.

Definition 2.2.1.3. [21], [27] Let $A = [a_{i,j}]$, $i = 1, 2 \ldots n$, $j = 1, 2 \ldots m$ be a given matrix and $\bar{g}$ a generator on $[-\infty, +\infty]$. Denote by $\bar{g}(A)$ the matrix $\bar{g}(A) = [\bar{g}(a_{i,j})]$, $i = 1, 2 \ldots n$, $j = 1, 2 \ldots m$. A generator $\bar{g}$ is rank-preserving operator if and only if $r(A) = r(\bar{g}(A))$ for every matrix A. Two generators $\bar{g}$ and $\bar{q}$ are rank-equivalence operators if and only if $r(\bar{g}(A)) = r(\bar{q}(A))$ for every matrix A.

Remark 2.2.1.4. [27] If $\bar{g}, \bar{q}$ are two rank-preserving operators, then $\bar{g}^{-1}$ and $\bar{g} \circ \bar{q}$ are rank-preserving operators.

Theorem 2.2.1.5. [27] The generators $\bar{g}$ and $\bar{q}$ are rank-equivalence operators on $[-\infty, +\infty]$ if and only if there exists a positive constant c so that $\bar{g} = c \cdot \bar{q}$.

If $\bar{q}$ is the identity function, then from Theorem 3.1 we directly get the theorem.

Theorem 2.2.1.6.. [27] The generator $\bar{g}$ is rank-preserving operator on the interval $[-\infty, +\infty]$ if and only if $\bar{g}$ is a linear function given by $\bar{g}(x) = c \cdot x$ with a positive constant c.

In this case when $\bar{q}$ is the identity function, we have the form of rank-preserving operator as the generator: $\bar{g}(x) = \bar{g}_{a,r}(x) = \bar{g}_{c,1}(x) = c \cdot x$ (i.e., constant $a = c$ and $r = 1$).

$$\boldsymbol{\bar{g}(x) = f_{a,lin}(x) = f\left(t^{(+,\cdot)(0,1,0,0)}(x, y)\right) = a \cdot x} \text{ (classes I, IV [9]).}$$

Teoremë 2.2.1.7. [21] Let $\bar{g}, \bar{q}$ be generators. These generations are in relation $\bar{g} = c \cdot \bar{q}$ then and only then the $\bar{g} - Transform$ of the linear sistems $A \cdot X = B$ and $A \cdot Y = B$, respectively pseudo-linear sistem $A \overline{\bigcirc}_{\bar{g}} X = B$ and $A \overline{\bigcirc}_{\bar{q}} Y = B$, are equivalent, i.e., the knowledge of X implies the knowledge of Y and vice versa, and $\bar{g}(X) = \bar{q}(Y)$.

Proof. The $\bar{g} - Transform$ of the linear sistem $A \cdot X = B$ is the pseudo-linear sistem, presented in the form of a $\bar{g} - linear\ system$ as below:

$$A \overline{\bigcirc}_{\bar{g}} X = B \Leftrightarrow \bar{g}^{-1}(\bar{g}(A) \cdot \bar{g}(X)) = \bar{g}^{-1}(\bar{g}(B)) \Leftrightarrow \bar{g}(A) \cdot \bar{g}(X) = \bar{g}(B) \Leftrightarrow$$
$$\Leftrightarrow \bar{g}(A) \cdot \underbrace{\bar{g}(X)}_{Z_1} = \bar{g}(B) \Leftrightarrow \bar{g}(A) \cdot Z_1 = \bar{g}(B).$$

The $\bar{q} - Transform$ of the linear sistem $A \cdot Y = B$ is the pseudo-linear sistem presented in the form of the $\bar{q} - linear\ system$: $A \overline{\bigcirc}_{\bar{q}} Y = B \Leftrightarrow \bar{q}^{-1}(\bar{q}(A) \cdot \bar{q}(Y)) = \bar{q}^{-1}(\bar{q}(B)) \Leftrightarrow \bar{q}(A) \cdot \bar{q}(Y) = \bar{q}(B) \Leftrightarrow \bar{q}(A) \cdot \underbrace{\bar{q}(Y)}_{Z_2} = \bar{q}(B) \Leftrightarrow \bar{q}(A) \cdot Z_2 = \bar{q}(B).$

From the equivalence of the pseudo-linear system $A \overline{\bigcirc}_{\bar{g}} X = B$, we reach the relation below: $A \overline{\bigcirc}_{\bar{g}} X = B \Leftrightarrow \bar{g}(A) \cdot Z_1 = \bar{g}(B) \Leftrightarrow c \cdot \bar{q}(A) \cdot Z_1 = c \cdot \bar{q}(B) \Leftrightarrow \bar{q}(A) \cdot Z_1 = \bar{q}(B)$ where $Z_1 = \bar{g}(X)$.

The both linear systems are equivalent and if they are solvable, we get $Z_1 = \bar{g}(X) = \bar{q}(Y) = Z_2$. For more details, we are expressing this equivalence obtained above in an explicit way continuing transformations:

$$A \overline{\bigcirc}_{\bar{g}} X = B \Leftrightarrow \bar{q}(A) \cdot Z_1 = \bar{q}(B) \Leftrightarrow \bar{q}^{-1}\left(\bar{q}(A) \cdot \bar{q}(\bar{q}^{-1}(Z_1))\right) = \bar{q}^{-1}(\bar{q}(B)) \Leftrightarrow A \overline{\bigcirc}_{\bar{q}} \bar{q}^{-1}(Z_1) =$$
$B \Leftrightarrow A \overline{\bigcirc}_{\bar{q}} \bar{q}^{-1}(\bar{g}(X)) = B \Leftrightarrow A \overline{\bigcirc}_{\bar{q}} Y = B$ where $Y = \bar{q}^{-1}(\bar{g}(X))$.

Remark. 2.2.1.8. Since from the conditions of the theorem we have $\bar{g} = c \cdot \bar{q}, c > 0$ and $\bar{g}(X) = \bar{q}(Y)$, then $Y = \bar{q}^{-1}(\bar{g}(X)) = \bar{g}^{-1}(c \cdot \bar{g}(X)) = \bar{g}^{-1}(\bar{g}(\bar{g}^{-1}(c) \cdot \bar{g}(X))) = \bar{g}^{-1}(c) \overline{\bigcirc}_{\bar{g}} X$.

Further, for the pseudo-multiplication $\overline{\bigcirc}_{\bar{q}}$, we can get the unity $e_{\bar{q}}$ by nga $e_{\bar{q}} = \bar{q}^{-1}(1) = \bar{g}^{-1}(c \cdot \bar{g}(1)) = \bar{g}^{-1}(c) \overline{\bigcirc}_{\bar{g}} \bar{g}(1)$. If $\bar{g}$ is normed generator $(\bar{g}(1) = 1)$, then the unity is in the form $e_{\bar{q}} = \bar{g}^{-1}(c) \overline{\bigcirc}_{\bar{g}} 1$ and

consequently we can obtain the pseudo-linear system $Y = X \overline{\odot}_{\bar{g}} \, e_{\bar{q}} = e_{\bar{q}} \overline{\odot}_{\bar{g}} \, X$.

2.2.2   Application in Probability Theory, Combinatorics and Algebra

Application 2.2.2.1. The $\bar{g} - Transform$ of Negation N

The $\bar{g} - Negation$ of the negation N [10], [28], $N(x) = 1 - x$ as a $\bar{g} - Transform$ (when $\bar{g}$ in general is continuous monotone strictly increasing unbounded odd function) is in form:

$$N_{\bar{g}}(x) = \bar{g}^{-1}(N(\bar{g}(x))) = \bar{g}^{-1}(1 - \bar{g}(x)) = \bar{g}^{-1}\left(\bar{g}(\bar{g}^{-1}(1)) - \bar{g}(x)\right) = \bar{g}^{-1}(1) \, \overline{\ominus}_{\bar{g}} \, x.$$

In summary, for each case of $\bar{g}$ and especially when it is normalized ($\bar{g}(1) = 1$),we present the form of $N_{\bar{g}}$:

$$\boldsymbol{N_{\bar{g}}(x)} = \begin{cases} \bar{g}^{-1}(1) \, \overline{\ominus}_{\bar{g}} \, x & for \; \bar{g} - whatever, random \\ 1 \, \overline{\ominus}_{\bar{g}} \, x & for \;\; \bar{g} - normed \end{cases}$$

$$\boldsymbol{N_{\bar{g}}(x)} = \begin{cases} \bar{g}^{-1}(1) \, \overline{\ominus}_{\bar{g}} \, t^{(+,\cdot)(0,1,0,0)}(x,y) & for \; \bar{g} - whatever, random \\ 1 \, \overline{\ominus}_{\bar{g}} \, t^{(+,\cdot)(0,1,0,0)}(x,y) & for \;\; \bar{g} - normed \end{cases}$$

$$\boldsymbol{N_{\bar{g}}(x)} = \boldsymbol{f_{\bar{g}}(x)} = \begin{cases} \bar{g}^{-1}(1) \, \overline{\ominus}_{\bar{g}} \, t_g^{\left((\oplus_{\bar{g}}, \overline{\odot}_{\bar{g}})\right)(0,1,0,0)}(x,y) & for \; \bar{g} - whatever, random \\ 1 \, \overline{\ominus}_{\bar{g}} \, t_{\bar{g}}^{\left((\oplus_{\bar{g}}, \overline{\odot}_{\bar{g}})\right)(0,1,0,0)}(x,y) & for \;\; \bar{g} - normed \end{cases}$$

Application 2.2.2.2. The $\bar{g} = \bar{g}_{a,r} - Transform$ of $\Delta - Pascal's \, Triangle$

In mathematics **Pascal's triangle** is a triangular array of the binomial coefficients that arises in Probability Theory, Combinatorics and Algebra. Here, we have taken into consideration the generalization of the $\Delta - Pascal's \, Triangle$ formula using a generator $\bar{g}_{a,r}$. For more, by the $\bar{g} = \bar{g}_{a,1} = \bar{g}_{1,1} - Transform$ we take $\Delta_{\bar{g}_{a,1}} = \Delta_{\bar{g}_{1,1}} = \Delta - Pascal's \, Triangle$ formula ($\bar{g}_{a,r} - normed$), as the classical one.

Table 3. The form of $\Delta - Pascal's \, Triangle$ as $\Delta_{\bar{g}_{a,r}} - Pascal's \, Triangle$ by $\bar{g} = \bar{g}_{a,r} - Transform$

$\Delta_{\bar{g}_{a,r}} - $Pascal's Triangle	$(0 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1)$ $(0 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 0)$ $(0 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 2) \;\; (2 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 0)$ $(0 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 3) \;\; (3 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 3) \;\; (3 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 0)$ $(0 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 4) \;\; (4 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 6) \;\; (6 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 4) \;\; (4 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 0)$
$\Delta_{\bar{g}_{a,r}} - $Pascal's Triangle ($\bar{g}_{a,r} - normed$) $\bar{g}_{a,r}^{-1}(a) = 1$ We are marking this number like this $(1 = 1^r)$ for a regularity of the presentation of the formula.	$\left(\bar{g}_{a,r}^{-1}(a)\right)$ $\left(\bar{g}_{a,r}^{-1}(a)\right) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; \left(\bar{g}_{a,r}^{-1}(a)\right)$ $\left(\bar{g}_{a,r}^{-1}(a)\right) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 2) \;\; (2 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; \left(\bar{g}_{a,r}^{-1}(a)\right)$ $\left(\bar{g}_{a,r}^{-1}(a)\right) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 3) \;\; (3 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 3) \;\; (3 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; \left(\bar{g}_{a,r}^{-1}(a)\right)$ $\left(\bar{g}_{a,r}^{-1}(a)\right) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 4) \;\; (4 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 6) \;\; (6 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 4) \;\; (4 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; \left(\bar{g}_{a,r}^{-1}(a)\right)$ … (1) $(1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1)$ $(1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 2) \;\; (2 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1)$ $(1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 3) \;\; (3 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 3) \;\; (3 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1)$ $(1) \;\; (1 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 4) \;\; (4 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 6) \;\; (6 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 4) \;\; (4 \, \overline{\oplus}_{\bar{g}_{a,r}} \, 1) \;\; (1)$ $\ddot{\imath}$ $(1) \;\; \left((1^r + 1^r)^{\frac{1}{r}}\right) \;\; (1)$ $(1) \;\; \left((1^r + 2^r)^{\frac{1}{r}}\right) \;\; \left((2^r + 1^r)^{\frac{1}{r}}\right) \;\; (1)$ $(1) \;\; \left((1^r + 3^r)^{\frac{1}{r}}\right) \;\; \left((3^r + 3^r)^{\frac{1}{r}}\right) \;\; \left((3^r + 1^r)^{\frac{1}{r}}\right) \;\; (1)$ $(1) \;\; \left((1^r + 4^r)^{\frac{1}{r}}\right) \;\; \left((4^r + 6^r)^{\frac{1}{r}}\right) \;\; \left((6^r + 4^r)^{\frac{1}{r}}\right) \;\; \left((4^r + 1^r)^{\frac{1}{r}}\right) \;\; (1)$

$\Delta_{\bar{g}_{a,1}} = \Delta_{\bar{g}_{1,1}} =$	(1)
$= \Delta - $ Pascal's Triangle $(\bar{g}_{a,r} - normed)$	(1) (2) (1) (1) (3) (3) (1) (1) (4) (6) (4) (1) (1) (5) (10) (10) (5) (1) ...

Newton's Binomial Formula $(x+y)^n = \sum_{i=1}^{n} C_n^i \cdot x^i \cdot y^{n-i}$ by $\bar{g} - Transform$ is expressed in the form bellow [10]:

$$((1+y)^n)_{\bar{g}} = f_{g-n,power}\left(\boldsymbol{t}_{\bar{g}}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})(0,0,1,1)}(x,y)\right) = \left(\left(\boldsymbol{t}_{\bar{g}}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})(0,0,1,1)}(x,y)\right)^n\right)_{\bar{g}}.$$

2.2.2.1 The $\bar{g} - Entropy$ and relation with $N_{\bar{g}}$

In *Information Theory*, through examples using the meaning of entropy as a logarithmic measure of the rate of information transfer in a given language or message, we present the relation between the meaning of $\bar{g} - entropy$ $\left(H_{\bar{g}-a,m,N_{\bar{g}-m}}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}\right)$ with the meaning of entropy $\left(H_{a,\ p,1-p}^{(+,\ \cdot)}\right)$ [9], [13], [25], [26] realized through the $\bar{g}_{a,r} - logarithmic\ function$, i.e. a $\bar{g} - Transformation$ [4], [8], [13], [23], [24] and $\bar{g} - Negation\ (N_{\bar{g}-m})$ [10], [15], [16], [18].

$$H_{\bar{g}-a,m,N_{\bar{g}-m}}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}(B) = \bar{g}^{-1}\left(H_{a,\ p,1-p}^{(+,\ \cdot)}(B)\right) = -h_{\bar{g}}\big(m(B_1)\big) - h_{\bar{g}}\big(1 \ominus m(B_2)\big) =$$

$$= -\bar{g}^{-1}\left(\big(p(B_1)\big)\cdot f\big(p(B_1)\big)\right) - \bar{g}^{-1}\left(\big(1-p(B_1)\big)\cdot f\big(1-p(B_2)\big)\right) =$$

$$= -\left(\bar{g}^{-1}\left(\big(p(B_1)\big)\cdot f\big(p(B_1)\big)\right) + \bar{g}^{-1}\left(\big(1-p(B_1)\big)\cdot f\big(1-p(B_2)\big)\right)\right) =$$

$$= -\left(\left(m(B_1)\ \overline{\odot}_{\bar{g}}\ f_g\big(m(B_1)\big)\right)\overline{\oplus}_{\bar{g}}\left(\left(\bar{g}^{-1}(1)\ \overline{\ominus}_{\bar{g}}\ m(B_2)\right)\overline{\odot}_{\bar{g}}\left(f_{\bar{g}}\left(\bar{g}^{-1}(1)\ \overline{\ominus}_{\bar{g}}\ m(B_2)\right)\right)\right)\right) =$$

$$=$$

$$-\left(\left(m(B_1)\ \overline{\odot}_{\bar{g}}\ f_{\bar{g}-a,log}\big(m(B_1)\big)\right)\overline{\overline{\oplus}}_{\bar{g}}\left(\left(\bar{g}^{-1}(1)\ \overline{\ominus}_{\bar{g}}\ m(B_2)\right)\overline{\odot}_{\bar{g}}\left(f_{\bar{g}-a,log}\left(\bar{g}^{-1}(1)\ \overline{\ominus}_{\bar{g}}\ m(B_2)\right)\right)\right)\right).$$

And, if the generator is $\bar{g} - normed$, we have the form:

$$H_{\bar{g}-a,m,N_{\bar{g}-m}}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})}(B) = -h_{\bar{g}}\big(m(B_1)\big) - h_{\bar{g}}\left(1\ \overline{\ominus}_{\bar{g}}\ m(B_2)\right) = -h_{\bar{g}}\big(m(B_1)\big) - h_{\bar{g}}\left(N_{\bar{g}-m}(B_2)\right) =$$

$$= -\left(h_{\bar{g}}\big(m(B_1)\big)\overline{\oplus}_{\bar{g}}h_{\bar{g}}\left(1\ \overline{\ominus}_{\bar{g}}\ m(B_2)\right)\right) = -\left(h_{\bar{g}}\big(m(B_1)\big)\overline{\oplus}_{\bar{g}}h_{\bar{g}}\left(N_{\bar{g}-m}(B_2)\right)\right).$$

2.2.3   Some Generalization in Euclidean Geometry

Generalizations and transformations by $\bar{g} - modification$ have been made for important formulas in Euclidean Geometry (generalization of the Pythagorean Theorem as $\bar{g}_{1,2} - Pythagorean\ Theorem$, coordinates of a point, distance formula etc.).

Based on the extended forms of $\bar{g} - calculus$, the special $\bar{g} - generators$ and the definition of the Parameterized Nonlinear/ Pseudo-Nonlinear Functions and their $\bar{g} - transforms\ (\boldsymbol{t}, \boldsymbol{t}_{\bar{g}} - function)$ [10], we present the relations between them by applying these modifications:

$$\boldsymbol{t}^{(+,\cdot)(a_1,a_2,a_3,a_4)}(\bar{g}(x),\bar{g}(y)) = a_1 \cdot \bar{g}(x)\cdot\bar{g}(y) + a_2 \cdot \bar{g}(x) + a_3 \cdot \bar{g}(y) + a_4,$$

$$\left(\boldsymbol{t}^{(+,\cdot)(a_1,a_2,a_3,a_4)}(f(x),f(y))\right)_{\bar{g}} = t_{\bar{g}}^{(\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}})\left(\bar{g}^{-1}(a_1),\bar{g}^{-1}(a_2),\bar{g}^{-1}(a_3),\bar{g}^{-1}(a_4)\right)}\left(f_{\bar{g}}(x),f_{\bar{g}}(y)\right).$$

For special $\bar{g} - generators$,
$\bar{g}(x) = \bar{g}_{a,r} = \bar{g}_{1,2}(x) = x^2\ (i.e., constant\ a = 1\ and\ r = 2)$,
$\bar{g}(x) = \bar{g}_{a,r} = \bar{g}_{a,2}(x) = a\cdot x^2\ (i.e., r = 2)$, we have the form of $\boldsymbol{t} - function$ as:

$$t^{(+,\cdot)(1,1,1,1)}\left(\bar{g}_{1,2}(x), \bar{g}_{1,2}(x)\right) = x^2 \cdot y^2 + x^2 + y^2 + 1,$$
$$t^{(+,\cdot)(1,1,1,1)}\left(\bar{g}_{a,2}(x), \bar{g}_{a,2}(x)\right) = a^2 \cdot x^2 \cdot y^2 + a \cdot x^2 + a \cdot y^2 + 1.$$

2.2.4 Generalization of the Pythagorean Equation $(c^2 = a^2 + b^2)$ as $\bar{g}_{1,2} - Pythagorean\ Equation$ and $t - Pythagorean\ Equation$

The Pythagorean Equation $c^2 = a^2 + b^2$ is express by $t - function$ in form:

$t^{(+,\cdot)(0,1,1,0,0)}\left(\bar{g}_{1,2}(a), \bar{g}_{1,2}(b), \bar{g}_{1,2}(c)\right) = \bar{g}_{1,2}(a) + \bar{g}_{1,2}(b) = a^2 + b^2.$
$t^{(+,\cdot)(0,0,0,1,0)}\left(\bar{g}_{1,2}(a), \bar{g}_{1,2}(b), \bar{g}_{1,2}(c)\right) = \bar{g}_{1,2}(c) = c^2.$
The Pythagorean equation ($t - Pythagorean\ Equation$):
- $t^{(+,\cdot)(0,1,1,0,0)}\left(\bar{g}_{1,2}(a), \bar{g}_{1,2}(b), \bar{g}_{1,2}(c)\right) = t^{(+,\cdot)(0,0,0,1,0)}\left(\bar{g}_{1,2}(a), \bar{g}_{1,2}(b), \bar{g}_{1,2}(c)\right),$
- $(\bar{g}_{1,2} - Pythagorean\ Equation)$: $\bar{g}_{1,2}(c) = \bar{g}_{1,2}(a) + \bar{g}_{1,2}(b).$
The hypotenuse **c** of right-angled triangle with sides **a, b** as $c_{\bar{g}_{1,2}}$:
- $(\bar{g}_{1,2} - hypotenuse)$: $c_{\bar{g}_{1,2}} = \bar{g}_{1,2}{}^{-1}\left(\bar{g}_{1,2}(a) + \bar{g}_{1,2}(b)\right) = a \,\overline{\oplus}_{\bar{g}_{1,2}}\, b$.

2.2.5 Generalization of the distance $d(M_1, M_2)$ as $\bar{g}_{1,2} - distance\ \left(d_{\bar{g}_{1,2}}(M_1, M_2)\right)$

For two points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ in a plain (2D, or the same we can use for 3D format), we take $\bar{g}_{1,2} - distance$:

$t^{(+,\cdot)(0,1,1,0)}\left(\bar{g}_{1,2}(x_2 - x_1), \bar{g}_{1,2}(y_2 - y_1)\right) = d(M_1, M_2) =$
$= \bar{g}_{1,2}(x_2 - x_1) + \bar{g}_{1,2}(y_2 - y_1) = (x_2 - x_1)^2 + (y_2 - y_1)^2.$
$g_{1,2}{}^{-1}\left(\bar{g}_{1,2}(x_2 - x_1) + \bar{g}_{1,2}(y_2 - y_1)\right) = (x_2 - x_1)\,\overline{\oplus}_{\bar{g}_{1,2}}\,(y_2 - y_1).$
$t_{\bar{g}_{1,2}}^{\left(\overline{\oplus}_{\bar{g}_{1,2}}, \overline{\ominus}_{\bar{g}_{1,2}}\right)(0,1,1,0)}(x_2 - x_1, y_2 - y_1) = d_{\bar{g}_{1,2}}(M_1, M_2) = (x_2 - x_1)\,\overline{\oplus}_{\bar{g}_{1,2}}\,(y_2 - y_1).$
Specific case, when $O(0,0)$ is the origin of coordinate system:
$t_{\bar{g}_{1,2}}^{\left(\overline{\oplus}_{\bar{g}_{1,2}}, \overline{\ominus}_{\bar{g}_{1,2}}\right)(0,1,1,0)}(x_M, y_M) = d_{\bar{g}_{1,2}}(O, M) = x_M\,\overline{\oplus}_{\bar{g}_{1,2}}\,y_M.$

2.2.6 Some Generalization in Trigonometry

2.2.6.1 Generalization of the Basic Formula of Trigonometry as the $\bar{g}_{1,2} - Basic\ Formula\ of\ Trigonometry$

From Basic Formula of Trigonometry to $\bar{g}_{1,2} - Formula$

$(cos\varphi)^2 + (sin\varphi)^2 = 1 \Leftrightarrow t^{(+,\cdot)(0,1,1,0)}\left(\bar{g}_{1,2}(cos\varphi), \bar{g}_{1,2}(sin\varphi)\right) = 1 \Leftrightarrow$
$\Leftrightarrow \bar{g}_{1,2}(1) = \bar{g}_{1,2}(cos\varphi) + \bar{g}_{1,2}(sin\varphi) \Leftrightarrow$
$\Leftrightarrow 1 = \bar{g}_{1,2}^{-1}\left(\bar{g}_{1,2}(cos\varphi) + \bar{g}_{1,2}(sin\varphi)\right) \Leftrightarrow$
$\Leftrightarrow 1 = cos\varphi\,\overline{\oplus}_{\bar{g}_{1,2}}\,sin\varphi.$
$\bar{g}_{1,2} - Basic\ Formula\ of\ Trigonometry$: $1 = cos\varphi\,\overline{\oplus}_{\bar{g}_{1,2}}\,sin\varphi.$

2.2.6.2 Generalization of trigonometric formulas, trig functions - sin, cos, tan, cot (for angles φ in the first quadrant)

$cos\varphi = \sqrt{1 - (sin\varphi)^2} = \bar{g}_{1,2}^{-1}\left(1 - \bar{g}_{1,2}(sin\varphi)\right) =$
$\qquad = \bar{g}_{1,2}^{-1}\left(\bar{g}_{1,2}(1) - \bar{g}_{1,2}(sin\varphi)\right) = 1\,\overline{\ominus}_{\bar{g}_{1,2}}\,sin\varphi.$
$sin\varphi = \sqrt{1 - (cos\varphi)^2} = \bar{g}_{1,2}^{-1}\left(1 - \bar{g}_{1,2}(cos\varphi)\right) =$
$\qquad = \bar{g}_{1,2}^{-1}\left(g_{1,2}(1) - \bar{g}_{1,2}(cos\varphi)\right) = 1\,\overline{\ominus}_{\bar{g}_{1,2}}\,cos\varphi.$

The relations below show the relationship between the trigonometric functions **sin, cos** as well as the expression depending on pseudo-Negation $N_{\bar{g}_{1,2}}$, $(\bar{g}_{1,2} - Negation)$:

$$cos\varphi = 1 \; \overline{\ominus}_{\bar{g}_{1,2}} \; sin\varphi = N_{\bar{g}_{1,2}}(sin\varphi),$$

$$sin\varphi = 1 \; \overline{\ominus}_{\bar{g}_{1,2}} \; cos\varphi = N_{\bar{g}_{1,2}}(cos\varphi).$$

$$tg\alpha = \frac{sin\alpha}{cos\alpha} \quad (tg\alpha)^2 = \frac{(sin\alpha)^2}{(cos\alpha)^2} \quad \bar{g}_{1,2}(tg\alpha) = \frac{\bar{g}_{1,2}(sin\alpha)}{\bar{g}_{1,2}(cos\alpha)},$$

$$tg\alpha = \bar{g}_{1,2}^{-1}\left(\frac{\bar{g}_{1,2}(sin\alpha)}{\bar{g}_{1,2}(cos\alpha)}\right) = sin\alpha \; \overline{\oslash}_{g_{1,2}} \; cos\alpha,$$

$$cot\alpha = \bar{g}_{1,2}^{-1}\left(\frac{\bar{g}_{1,2}(cos\alpha)}{\bar{g}_{1,2}(sin\alpha)}\right) = cos\alpha \; \overline{\oslash}_{g_{1,2}} \; sin\alpha,$$

$$tg\alpha = \frac{1}{cotg\alpha} \quad \bar{g}_{1,2}(tg\alpha) = \frac{\bar{g}_{1,2}(1)}{\bar{g}_{1,2}(cotg\alpha)} \text{ so,}$$

$$tg\alpha = \bar{g}_{1,2}^{-1}\left(\frac{\bar{g}_{1,2}(1)}{\bar{g}_{1,2}(cotg\alpha)}\right) = 1 \; \overline{\oslash}_{g_{1,2}} \; cot\alpha = N_{\bar{g}_{1,2}}(cot\alpha).$$

In the same form, the relations below show the relationship between the trigonometric functions **tan, cot** as well as the expression depending on pseudo-Negation $N_{\bar{g}_{1,2}}$, $(\bar{g}_{1,2} - Negation)$:

$$tg\alpha = 1 \; \overline{\oslash}_{g_{1,2}} \; cot\alpha = N_{\bar{g}_{1,2}}(cot\alpha),$$

$$cot\alpha = 1 \; \overline{\oslash}_{g_{1,2}} \; tg\alpha = N_{\bar{g}_{1,2}}(tg\alpha).$$

For each trigonometric function - **sin, cos, tan, cot,** we can apply the $\bar{g}_{1,2} - Transform$ and the four equations are simple forms:

$$(sin\alpha)_{\bar{g}} = sin_{\bar{g}_{1,2}}\alpha = \bar{g}_{1,2}^{-1}\left(sin\left(\bar{g}_{1,2}(\alpha)\right)\right) = \sqrt{sin\alpha^2},$$

$$(cos\alpha)_{\bar{g}} = cos_{\bar{g}_{1,2}}\alpha = \bar{g}_{1,2}^{-1}\left(cos\left(\bar{g}_{1,2}(\alpha)\right)\right) = \sqrt{cos\alpha^2},$$

$$(tg\alpha)_{\bar{g}} = tg_{\bar{g}_{1,2}}\alpha = \bar{g}_{1,2}^{-1}\left(tg\left(\bar{g}_{1,2}(\alpha)\right)\right) = \sqrt{tg\alpha^2},$$

$$(cot\alpha)_{\bar{g}} = cot_{\bar{g}_{1,2}}\alpha = \bar{g}_{1,2}^{-1}\left(cot\left(\bar{g}_{1,2}(\alpha)\right)\right) = \sqrt{cot\alpha^2}.$$

For each trigonometric function - **sin, cos,** we can use the relationship with the $\bar{g}_{1,2} - Negation,$ thus we obtain the following equations:

$$N_{\bar{g}_{1,2}}\left(cos_{\bar{g}_{1,2}}\alpha\right) = \bar{g}_{1,2}^{-1}\left(1 - \bar{g}_{1,2}\left(cos_{\bar{g}_{1,2}}\alpha\right)\right) = \bar{g}_{1,2}^{-1}\left(1 - \bar{g}_{1,2}\left(\bar{g}_{1,2}^{-1}\left(cos\left(\bar{g}_{1,2}(\alpha)\right)\right)\right)\right) =$$

$$= \bar{g}_{1,2}^{-1}\left(1 - cos\left(\bar{g}_{1,2}(\alpha)\right)\right),$$

$$N_{\bar{g}_{1,2}}\left(sin_{\bar{g}_{1,2}}\alpha\right) = \bar{g}_{1,2}^{-1}\left(1 - \bar{g}_{1,2}\left(sin_{\bar{g}_{1,2}}\alpha\right)\right) = \bar{g}_{1,2}^{-1}\left(1 - \bar{g}_{1,2}\left(\bar{g}_{1,2}^{-1}\left(sin\left(\bar{g}_{1,2}(\alpha)\right)\right)\right)\right) =$$

$$= \bar{g}_{1,2}^{-1}\left(1 - sin\left(\bar{g}_{1,2}(\alpha)\right)\right).$$

We can present these relationships in simple forms:

$$N_{\bar{g}_{1,2}}\left(cos_{\bar{g}_{1,2}}\alpha\right) = \sqrt{1 - cos\alpha^2},$$

$$N_{\bar{g}_{1,2}}\left(sin_{\bar{g}_{1,2}}\alpha\right) = \sqrt{1 - sin\alpha^2}.$$

3. Results and Discussions

The study method used as well as the application of Pseudo-Analysis knowledge in some areas examined in this study (but not only) have led us to some important results, discussions regarding the treatments made in the paper as well as the study perspective of our work further research.

- Based on the extended forms of $\bar{g} - calculus$, $\bar{g} - generators$ and the definition of the Parameterized Nonlinear Functions (t) and Parameterized Pseudo-Nonlinear Functions $(t_{\bar{g}})$ we present the relations between them. These relationships helped us to carry out the study related to the generalizations of Pseudo-Analysis in the fields studied according to the respective applications.

- Pseudo-Analysis and Pseudo-Linear Algebra are fields with many close connections, especially the $\bar{g} - Transform$ of the linear sistems to $\bar{g} - linear\ sistem$ open us a good line for further study about matrix. In our future research, the cases where pseudo-arithmetic operations are idempotent will be of interest, extending the study to applications of Idempotent Pseudo-Analysis.

- The use of some special $\bar{g}_{a,r} - generators$, the role of them in construction of $\bar{g} - function$ and $\bar{g} - Negation$ have led us to important relationships, giving us the opportunity to study some interesting applications in different fields.

- The proces of generalizations and modifications by $\bar{g} - transforms$ of composition of some real continuous parameterized functions with other special functions, bring some connections of Pseudo-Analysis with other fields such as Pseudo-Linear Algebra, Information Theory, Probability, Combinatorics, Geometry, Trigonometry, Elementary Algebra and other areas of pure mathematics connected with combinatorial problems.

- $\Delta - Pascal's\ Triangle$ formula is treated with input (0) from two sides (left-right) of the triangle $\left(0 \overline{\oplus}_{\bar{g}_{a,r}} 1\right) = \left(1 \overline{\oplus}_{\bar{g}_{a,r}} 0\right) = \left(\bar{g}_{a,r}^{\ -1}(a)\right)$ and $(\bar{g}_{a,r} - normed)\ \bar{g}_{a,r}^{\ -1}(a) = 1$. By the $\bar{g} = \bar{g}_{a,1} = \bar{g}_{1,1} - Transform$ of the $\Delta - Pascal's\ Triangle$ formula, we take $\Delta_{\bar{g}_{a,1}} = \Delta_{\bar{g}_{1,1}} = \Delta - Pascal's\ Triangle$ formula $(\bar{g}_{a,r} - normed)$, as the classical one.

- In *Information Theory*, through examples using the meaning of entropy as a logarithmic measure of the rate of information transfer in a given language or message, we present the relation between the meaning of $\bar{g} - entropy$ $\left(H^{(\oplus_{\bar{g}}, \odot_{\bar{g}})}_{\bar{g}-a,m,N_{\bar{g}-m}}\right)$ with the meaning of entropy $\left(H^{(+,\ \cdot)}_{a,\ p,1-p}\right)$, realized through the $\bar{g}_{a,r} - logarithmic\ function$, i.e. a $\bar{g} - Transformation$ and $\bar{g} - Negation$ $\left(N_{\bar{g}-m}\right)$.

- The interesting relations between the trigonometric functions *sin, cos, tan, cot* performed by $\bar{g} - Transform$ as well as the expression depending on Pseudo-Negation $N_{\bar{g}_{1,2}}$, show the role of Generated Pseudo-Analysis as a generalization of the Classical Analysis and connections with other fields. As a result, we have created an expanded table of trigonometric function relationships which will be further enriched.

- The classes (**I, IV**) of real continuous parameterized nonlinear and pseudo-nonlinear functions showed their importance and role in these connections between fields along with interesting applications, as well as during transformations.

4. Conclusions

- The interesting relations performed by $\bar{g} - Transform$ for some important function and formula shown through applications, as well as the expression depending on Pseudo-Negation $N_{\bar{g}_{a,r}}$, highlight the role of Generated Pseudo-Analysis as a generalization of the Classical Analysis.

- In the focus of our research work will be more investigations and developments on classes of the Nonlinear Functional Equations $\left(NL.F.Eq. - f_{\bar{g}}\right)$ the Pseudo-Nonlinear Functional Equations $\left(PNL.F.Eq. - f_{\bar{g}} \circ t_{\bar{g}}\right)$ modified by $\bar{g} - transform$. (classes **I, II, III, IV**) which are also conditioned by other generators of study importance.

- There are also many developments in these fields as well as in others, which can be addressed in our study work and which will be presented further (treatments that could not be included in this work).

- The theory of Pseudo-Analysis puts us in front of new perspectives for research and more interesting applications of Generated and Idempotent Pseudo-Analysis in other fields.

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