Homomorphism in Weakly $\Gamma$-Ring

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Abstract

Many algebraic structures have been defined so far. One of them, is that of $\Gamma$-ring, which is a generalization of ring. Weakening some of the conditions of the definition of $\Gamma$-ring, it has also been defined the concept of weakly $\Gamma$-ring. An important and well-known concept for every algebraic structure is homomorphism. In this paper, the concept of homomorphism in weakly $\Gamma$-ring is introduced. Further, some simple results analogous to the theory of rings, related to this concept are extended.

Keywords: $\Gamma$-semigroup, $\Gamma$-ring, weakly $\Gamma$-ring, homomorphism

1. Introduction

The concept of $\Gamma$-ring, which is a generalization of the concept of ring, was first defined by Nobusawa in [1].

Barnes, in [2], weakened some of the conditions of the definition of Nobusawa, and defined those that he called $\Gamma$-rings, naming $\Gamma$-rings defined in [1], as $\Gamma$-rings of Nobusawa.

Based to the definition of Nobusawa’s $\Gamma$-ring, Sen in [3], defined $\Gamma$-semigroup that is called $\Gamma$-semigroup of Sen. Sen and Saha in [4], defined a generalization of $\Gamma$-semigroup of Sen, which is called a $\Gamma$-semigroup. The concept of $\Gamma$-semigroup may be obtained by that of $\Gamma$-ring, by extracting addition.

Petro and Sema, in [5], weakened further the conditions of the definition of Barnes and defined those that they called weakly $\Gamma$-rings.

An important concept for every algebraic structure is homomorphism. Thus, it is eligible extending this concept to weakly $\Gamma$-rings.

In this paper, homomorphism to weakly $\Gamma$-rings is introduced and some simple results of rings to $\Gamma$-rings, which are related to this concept, are extended.

2. Materials and Methods

Here we give some notions and present some auxiliary results that will be used throughout the paper.

Let $M$ and $\Gamma$ be two nonempty sets. Any map from $M \times \Gamma \times M$ to $M$ is called a $\Gamma$-multiplication on $M$ and is denoted by $(\cdot)_{\Gamma}$. The result of this $\Gamma$-multiplication for each $a, b \in M$ and each $\gamma \in \Gamma$, is denoted by $a\gamma b$.

The concept of $\Gamma$-ring, which is a generalization of the concept of ring, was first defined by Nobusawa in [1], as follows:
Definition 2.1. [1] Let $M$ be an additive group with elements $a, b, c, \ldots$ and $\Gamma$ another additive group with elements $\alpha, \beta, \gamma, \ldots$. Assume that $a \alpha b$ is defined as an element of $M$ and $\alpha \alpha b$ is defined as an element of $\Gamma$ for each $a, b, \alpha$ and $\beta$. If the products satisfy the following conditions:

1. $(a_1 + a_2) \alpha b = a_1 \alpha b + a_2 \alpha b$, $a \alpha (b_1 + b_2) = a \alpha b_1 + a \alpha b_2$,
2. $(a_1 + a_2) \alpha b = a_1 \alpha b + a_2 \alpha b$.

Then $M$ is called a $\Gamma$-ring.

An ordinary ring $(A, +, \cdot)$ may turn into a $\Gamma$-ring, if we get $M$ and $\Gamma$ to be equal to $A$.

Barnes, in [2], weakened some of the conditions of Nobusawa, by calling $\Gamma$-rings of Nobusawa the ones defined as above and simply $\Gamma$-rings those that he defined himself.

Definition 2.2. [2] Every ordered five-tuple $(M, \Gamma, +, \oplus, \cdot)$, where $M$, $\Gamma$ are sets, $+$ is an addition on $M$, $\oplus$ addition on $\Gamma$, $\cdot$ is a $\Gamma$-multiplication on $M$, such that:

1. $(M, +)$ is an abelian group.
2. $(\Gamma, \oplus)$ is an abelian group.
3. $\forall (a, b, c, \alpha, \beta) \in M \times \Gamma$, $(a \alpha b) \oplus c = a \alpha (b \oplus c) \alpha c$.
4. $\forall (a, b, c, \gamma) \in M \times \Gamma$, $a \gamma (b + c) = a \gamma b + a \gamma c$.
5. $\forall (a, b, c, \gamma) \in M \times \Gamma$, $a \gamma (b + c) = a \gamma b + a \gamma c$.
6. $\forall (a, b, \alpha, \beta) \in M \times \Gamma$, $a \alpha (b \oplus \beta) = a \alpha b + a \alpha \beta$, is called $\Gamma$-ring (of Barnes).

Sen and Saha in [4], defined $\Gamma$-semigroups, which may be obtained by the definition of $\Gamma$-rings, by avoiding the additions:

Definition 2.3. [4] Every ordered pair $(M, \cdot)$, where $M$ and $\Gamma$ are two nonempty sets and $\cdot$ is a $\Gamma$-multiplication on $M$, such that:

1. $(M, +)$ is an abelian group.
2. $(\Gamma, \oplus)$ is an abelian group.
3. $\forall (a, b, c, \alpha, \beta) \in M \times \Gamma$, $(a \alpha b) \oplus c = a \alpha (b \oplus c) \alpha c$.
4. $\forall (a, b, c, \gamma) \in M \times \Gamma$, $a \gamma (b + c) = a \gamma b + a \gamma c$.
5. $\forall (a, b, c, \gamma) \in M \times \Gamma$, $a \gamma (b + c) = a \gamma b + a \gamma c$.
6. $\forall (a, b, \alpha, \beta) \in M \times \Gamma$, $a \alpha (b \oplus \beta) = a \alpha b + a \alpha \beta$, is called $\Gamma$-ring (of Barnes).

Petro and Sema in [5], weakened further the conditions of the definition of $\Gamma$-rings (of Barnes), by defining weakly $\Gamma$-rings, as follows:

Definition 2.4. [5] Every ordered triple $(M, +, \cdot)$, where $M$ and $\Gamma$ are two nonempty sets, $+$ is an addition on $M$, and $\cdot$ is a $\Gamma$-multiplication on $M$, such that:

1) $(M, +)$ is an abelian group.
2) $(\Gamma, \oplus)$ is an abelian group.
3) $\forall (a, b, c, \alpha, \beta) \in M \times \Gamma$, $(a \alpha b) \oplus c = a \alpha (b \oplus c) \alpha c$.
4) $\forall (a, b, c, \gamma) \in M \times \Gamma$, $a \gamma (b + c) = a \gamma b + a \gamma c$.
5) $\forall (a, b, c, \gamma) \in M \times \Gamma$, $a \gamma (b + c) = a \gamma b + a \gamma c$.
6) $\forall (a, b, \alpha, \beta) \in M \times \Gamma$, $a \alpha (b \oplus \beta) = a \alpha b + a \alpha \beta$, is called weakly $\Gamma$-ring.

We notice that plain rings, $\Gamma$-rings of Nobusawa and $\Gamma$-rings of Barnes, are weakly $\Gamma$-rings, but the converse is not true.

Saha and Seth in [6] have introduced the concept of homomorphism between two $\Gamma$-semigroups, as follows:

Definition 2.5. [6] Let $(M, \cdot)$ be a $\Gamma$-semigroup and $(M, \cdot)$ be a $\Gamma_1$-semigroup. A pair of mappings $(h_1, h_2)$, where $h_1: M \to M$, $h_2: B \to B$, such that $h_1(a \cdot b) = h_1(a) h_2(b)$, called a homomorphism of $(M, \cdot)$ to $(M, \cdot)$.

Let $(M, +, \cdot)$ be a weakly $\Gamma$-ring. Every nonempty subset $T$ of $M$, such that $(T, +)$ is a subgroup of $(M, +)$ and $a \cdot b \in T$, for each $(a, b) \in T$ and $\gamma \in \Gamma$, is called sub$\Gamma$-ring of $M$.

Let $M$ be a weakly $\Gamma$-ring and $A$, $B$ two nonempty subsets of $M$. Define:

$\Gamma^T = \{ \sum_{i=1}^n a_i \gamma_i b_i : a_i \in A, b_i \in B, \gamma_i \in \Gamma \}$ for each $i = 1, 2, \ldots, n; n \in \mathbb{N}$. 

Every subgroup $R[L]$ of the group $(M, +)$, such that:

$R \Gamma^M \subseteq R \subseteq \mathbb{L}$, 

is called right [left] ideal of the weakly $\Gamma$-ring $(M, +, \cdot)$. Every subgroup $I$ of the group $(M, +)$, such that:

$\Gamma^I \subseteq \mathbb{L}$, 

is called right [left] ideal of the weakly $\Gamma$-ring $(M, +, \cdot)$.
is called ideal of the weakly $Γ$-ring $(M, +, (\cdot)_r)$.

Thus, $I$ is an ideal of the weakly $Γ$-ring $M$, only if it is a left ideal and a right ideal of $M$ simultaneously.

3. Conclusions

In this section, basing on what is given above, mixing them, some new results are given.

**Definition 3.1.** Let $(M, +, (\cdot)_r)$ be a weakly $Γ$-ring and $(M', +, (\cdot)_{r'})$ be a weakly $Γ'$-ring. Every ordered pair of mappings $H = (h_1, h_2)$, where $h_1 : M \to M'$ and $h_2 : Γ \to Γ'$, such that:

1) \[ ∀(a, b) ∈ M^2, h_1(a + b) = h_1(a) + h_1(b). \]
2) \[ ∀(a, α, b) ∈ M × Γ × M, h_1(αab) = h_1(α)h_2(α)h_1(b), \]

is called a homomorphism of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$.

It is obvious that every homomorphism of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$, is an ordered pair of mappings $(h_1, h_2)$, where $h_1$ is a homomorphism of the additive group $(M, +)$ of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the additive group $(M', +)$ of the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$, whereas $h_2$ is a homomorphism of the $Γ$-semigroup $(M, (\cdot)_r)$ to the $Γ'$-semigroup $(M', (\cdot)_{r'})$ of the $Γ'$-multiplication of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the $Γ'$-ring $(M', +, (\cdot)_{r'})$.

If both mappings $h_1, h_2$ are injective (one-to-one), the homomorphism $H = (h_1, h_2)$ is called monomorphism and $(M, Γ)$ is called monomorph to $(M', Γ')$. If these mappings are both surjective (onto), $H$ is called epimorphism and $(M, Γ)$ is called epiomorph to $(M', Γ')$.

**Definition 3.2.** Let $H = (h_1, h_2)$ be a homomorphism of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$. The kernel of the homomorphism $h_1$ of the additive group $(M, +)$ to the additive group $(M', +)$, is called kernel of the homomorphism $H$ and will be denoted by $KerH$. Thus:

\[ KerH = \{ x ∈ M : h_1(x) = 0_M \}. \]

The kernel of the homomorphism $h_1$, is called image of the homomorphism $H = (h_1, h_2)$, and will be denoted by $ImH$. Thus:

\[ ImH = \{ h_1(x) ∈ M' : x ∈ M \} = \{ x' ∈ M' : ∃ x ∈ M, h_1(x) = x' \}. \]

**Proposition 3.3.** For every homomorphism $H = (h_1, h_2)$ of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$, the kernel $KerH$ is an ideal of the weakly $Γ$-ring $(M, +, (\cdot)_r)$, whereas, when $h_2$ is a surjective mapping, $ImH$ is a sub-$Γ'$-ring of the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$.

**Proof.** $KerH$ is a subgroup of the additive group of the weakly $Γ$-ring $(M, +, (\cdot)_r)$. For each element $α$ of $M$, for each element $b$ of $KerH$ and for each element $γ$ of $Γ'$, the following hold:

\[ h_1(αb) = h_1(α)h_2(γ)h_1(b) = 0h_2(γ)h_1(b) = 0, \]
\[ h_1(bαa) = h_1(b)h_2(γ)h_1(α) = h_1(b)h_2(γ)0 = 0, \]

which show that $KerH$ is an ideal of the weakly $Γ$-ring $(M, +, (\cdot)_r)$.

Let $a'$, $b'$ be two elements of $ImH$ and $γ'$ an arbitrary element of $Γ'$. Then, there exist the elements $a, b ∈ M$ and the element $γ ∈ Γ$, such that:

\[ a' = h_1(a), b' = h_1(b), γ' = h_2(γ). \]

The following equalities hold:

\[ a'γ'b' = h_1(a)h_2(γ)h_1(b) = h_1(αyb), \]

which show that $ImH$ is a sub-$Γ'$-ring of the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$, since $ImH$ is a subgroup of the group $(M', +)$.

Let $H = (h_1, h_2)$ be a homomorphism of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$, and $B'$ a subset of $M'$. Denote

\[ H^{-1}(B') = \{ x ∈ M : h_1(x) ∈ B' \}. \]

The subset $H^{-1}(B')$ of $M$, will be called inverse image of $B'$.

**Proposition 3.4.** Let $H = (h_1, h_2)$ be an epimorphism of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ to the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$ with kernel, the ideal $I$ of $(M, +, (\cdot)_r)$. Then, a nonempty subset $B'$ of $M'$ is an ideal of $(M', +, (\cdot)_{r'})$ if and only if

\[ H^{-1}(B') = B \]

is an ideal of the weakly $Γ$-ring $(M, +, (\cdot)_r)$ that contains the ideal $I$.

**Proof.** Assume that the nonempty subset $B'$ of $M'$ is an ideal of the weakly $Γ'$-ring $(M', +, (\cdot)_{r'})$. Let $α$ be an
ordinary element of $M$, $b$ an ordinary element of $B = H^{-1}(B')$ and $\gamma$ an ordinary element of $\Gamma$. Since:

$$h_1(ab) = h_1(a)h_2(b)h_3(b) \in M'\cap B' \subseteq B',$$
$$h_2(ba) = h_1(b)h_2(\gamma)h_3(a) \in B'\cap M' \subseteq B',$$

the elements $ab\gamma, b\gamma a$ belong to the subset $B$ and consequently $H^{-1}(B') = B$ is an ideal of the weakly $\Gamma$-ring $(M, +, (\cdot)_{\Gamma})$, since $H^{-1}(B')$ is a subgroup of the additive group $(M, +)$ of the weakly $\Gamma$-ring $(M, +, (\cdot)_{\Gamma})$.

The ideal $H^{-1}(B') = B$ contains the ideal $I$ of the epimorphism $H$; since we have

$$\forall x \in M, x \in I \Rightarrow h_2(x) = 0 \in B'.$$

Conversely, suppose that $H^{-1}(B') = B$ is an ideal of the weakly $\Gamma$-ring $(M, +, (\cdot)_{\Gamma})$, that contains the ideal $I$.

Let $\alpha'$ be an element of $M'$, $b'$ an element of $B'$ and $\gamma'$ an element of $\Gamma'$. There exist the elements $a \in M, b \in H^{-1}(B') = B$ and $\gamma \in \Gamma$, such that:

$$a' = h_1(a), \quad b' = h_1(b), \quad \gamma' = h_2(\gamma).$$

The following equalities hold:

$$a' - b' = h_1(a) - h_1(b) = h_1(a - b),$$
$$a'\gamma'b' = h_1(a)h_2(\gamma)h_3(b) = h_2(ab),$$
$$b'\gamma'a' = h_1(b)h_2(\gamma)h_3(a) = h_2(b\gamma a),$$

that show that $B'$ is an ideal of the weakly $\Gamma'$-ring $(M', +, (\cdot)_{\Gamma'})$, since $B' \neq \emptyset$, because $I \subseteq H^{-1}(B') = B$, and $H^{-1}(B') = B$ is an ideal of $(M, +, (\cdot)_{\Gamma})$, that guarantees us that $a-b, ab, b\gamma a$ are elements of $H^{-1}(B') = B$. ■

References