Some Propositions About Inverse Semigroups

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Received: 19 February 2024 / Accepted: 15 March 2024 / Published: 23 March 2024

Doi: 10.56345/ijrdv11n118

Abstract

Inverse semigroups are an important subclass of the class of semigroups. An important role in the properties of the semigroups in general and in the class of inverse semigroups in particular is played by the set of their idempotents, which we denote by E or ES for a semigroup S. In the first part of this paper we will show some propositions about left unitary, right unitary, unitary subsets and closed subsets in the inverse semigroups. We will also see some properties of the maximum idempotent - separating congruence in an inverse semigroup S and its connection with Clifford's semigroups.

Keywords: *inverse, normal, idempotent, congruence*

1. Introduction

Semigroups are one of the basic algebraic structures and among the most important in algebra, from which is given the meaning of all other algebraic structures. One important subclass of the class of semigroups is the class of inverse semigroups, whose structure is closer to that of the group. Therefore, the purpose of this article is to investigate how in certain circumstances of inverse semigroups and different properties that the set of its idempotents can satisfied, their structure satisfies other properties that bring them closer and closer to the structure of the group.

1.1 Left, right unitary and closed subsets

In the structure of semigroups there are some elements *a* for which there is at least one element *x* from this semigroup such that $a = axa$. Such element *a* is called regular [8] and the semigroups, each element of which is regular, are called regular semigroups [8]. A subclass of the class of regular semigroups is the class of inverse semigroups [6], [7], which have the property that for each element a, there is only one element a' such that $a = aa'a$ and $a' = a'aa'$. The elements a dhe a' are called inverses of each other. The notions left unitary, right unitary and unitary subset of a semigroup *S* are introduce from P.Dubreil in [1] as below:

***Definition 2.1*:** [1] The subset A of the subsemigroup S is called:
left unitary if $(\forall a \in A)(\forall s \in S)(as \in A \Longrightarrow s \in A)$,
right unitary if $(\forall a \in A)(\forall s \in S)(sa \in A \Longrightarrow s \in A)$,
and unitary if it is both left and right unitary.

The definition of the full subsemigroup of an inverse semigroup S is given from J.M.Howie in [2] in the sense that it contais all the idempotents of S. While, the relation "$\leq$" on an iverse semigroup was introduced by Vagner in [7] and it was continued later by J.M.Howie in [2]. So we have these definitions:

Definition 2.2: [2] A subsemigroup T of an inverse semigroup S is called full if T contains all the idempotents of S.

Definition 2.3: [2], [7] If a,b are two elements of an inversive subsemigroup S we will write $a \leq b$ if there exists an idempotent e in S such that $a = eb$.

Schein (1962) in [3] has introduced for the first time the meanings of the closure of a subset and the closed subset in an inverse semigroup S. Later, in 1967 these meanings are used from Clifford and Preston [4]. We are giving them by the definition below:

Definition 2.4: [3], [4] If H is an arbitrary subset of an inverse semigroup S, then we shall define the closure $H\omega$ of H by $H\omega = \{x \in S : \exists h \in H, h \leq x\}$. We shall say that H is closed if $H\omega = H$.

In the propositions 2.1, 2.2, 2.3 and 2.4 we will sow some relations between left unitary, right unitary and unitary subsets, full subsemigroups, closed subsemigroups and closure of a subset in an inverse semigroup S.

Proposition 2.1: If U is a full subsemigroup and left unitary of the inverse semigroup S, then U is an inverse subsemigroup of S.

Proof: Since U is a full subsemigroup of S it follows that all the idempotents of S belong to U, i.e. $E^S \subset U$. So, if $u \in U$, then uu' is idempotent that means $uu' \in U$. In the other hand U is a left unitary set of the semigroup S, so we will have $uu' \in U \wedge u \in U \Rightarrow u' \in U$. So the subsemigroup U of S, besides its element u, contains also the invers element u'. So we conclude that U is an inverse subsemigroup of the inverse semigroup S.

Proposition 2.2: If U is a left unitary subset and inverse subsemigroup of the inverse semiogroup S, then U is unitary subset of S, i.e.is left and right subset of S.

Proof: Since U is a left unitary subset of S, it is remain to show that U is also a right unitary subset of S, that means we must show that $(\forall u \in U)(\forall s \in S)(su \in U \Longrightarrow s \in U)$. First of all, since U is an inverse semigroup we will have that $(\forall u \in U)(u' \in U)$. So we have also:

$(\forall u \in U)(\forall s \in S)[su \in U \Rightarrow (su)' \in U \Rightarrow u's' \in U]$

because *U* is an inverse subsemigroup of the semigroup *S*. But *U* is a left unitary subset of *S*, hence we will have:

$(\forall u \in U)(\forall s \in S)(u' \in U \wedge u's' \in U \Rightarrow s' \in U \Rightarrow s \in U)$

so we have shown also that *U* is a right unitary subset of *S*, that means it is a unitary subset of the semigroup S.

Proposition 2.3: *If U is an inverse subsemigroup of the inverse semiogroup S, then U is unitary subset of S, if and only if U a closed subset of the semigroup S.*

Proof: First let be *U* an inverse subsemigroup of *S* and suppose that *U* is also an unitary subset of *S*. We must prove that *U* a closed subset of the semigroup *S,* i.e. $U = U\omega$. Due to show this we need to prove $U\omega \subset U$ because, from the definition of the $U\omega$, it is evident that $U \subset U\omega$. Indeed,

$x \in U\omega \Rightarrow (\exists u \in U)(u \leq x) \Rightarrow u = ex$,

where *e* is an idempotent of *S*. We also have:

$u \in U \Rightarrow ex \in U \Rightarrow e^2x \in U \Rightarrow e(ex) \in U.$

So, from $ex \in U$ and $e(ex) \in U$ we obtain that $e \in U$, because U is unitary subset of *S* that means it is also right unitary. Again from the fact that U is a right unitary subset of *S* and from $ex \in U$ and $e \in U$ we will have $x \in U$. So we have shown that $U\omega \subset U$ that means $U = U\omega$, i.e. U is closed subset of *S*.

Now we suppose that U is closed subset of *S* and must prove that U is an unitary subset of *S*. Firstly we will show that U is a right unitary subset of *S*. Let be $u \in U$ and $s \in S$ any two elements such that $su \in U$. We must prove that $s \in U$ that means U will be a right unitary subset of *S*. We also can see that:

$u \in U \Rightarrow u' \in U \Rightarrow uu' \in U$ or $uu' = e \in U$.

Now, from $u' \in U$ and $su \in U$ we obtain:

$(su)u' \in U \Rightarrow s(uu') \in U \Rightarrow se \in U$

Let denote $u_1 = se$. This means that $u_1 \in U$ and $u_1 \leq s$, so $s \in U\omega$, furthermore $s \in U$ because U is closed i.e. $U = U\omega$. So we have shown that U is a right unitary subset of *S*. Secondly we will show that U is a left unitary subset of *S*. Let be $u \in U$ and $s \in S$ any two elements such that $us \in U$. We must prove that $s \in U$ that means U will be a left unitary subset of *S*. We also can see that:

$u \in U \Rightarrow u' \in U \Rightarrow u'u \in U$ or $u'u = e \in U$

Now, from $u' \in U$ and $us \in U$ we obtain:

$u'(us) \in U \Rightarrow (u'u)s \in U \Rightarrow es \in U$

Let denote $u_2 = es$. This means that $u_2 \in U$ and $u_2 \leq s$, so $s \in U\omega$, furthermore $s \in U$ because U is closed i.e. $U = U\omega$. So we have shown that U is a left unitary subset of *S*. So we have proved that U is an unitary subset of *S*.

Proposition 2.4: *If K is a subset of the inverse semiogroup S and $s \in S$ is a random element, then the closure of the set Ks is equal to the closure of the set $((K\omega)s)$.*

Proof: We must to prove $(Ks)\omega = ((K\omega)s)\omega$. First, from the definition of the closure, we have that $K \subseteq K\omega$ that means $Ks \subseteq (K\omega)s$ and from that we obtain $(Ks)\omega \subseteq ((K\omega)s)\omega$. So it is remain to show that $(Ks)\omega \supseteq ((K\omega)s)\omega$. Due to show this, let be $x \in ((K\omega)s)\omega$ and we will have:

$x \in ((K\omega)s)\omega \Longrightarrow [\exists y \in (K\omega)s](y \leq x) \Longrightarrow (\exists e \in E)(y = ex)$

where *E* is the set of idempotents of *S*. On the other hand,

$y \in (K\omega)s \Longrightarrow y = k_1 s$ and $k_1 \in K\omega$,

so we have:

$k_1 \in K\omega \Longrightarrow (\exists k \in K)(k \leq k_1) \Longrightarrow k = e_1 k_1$

where e_1 is also an idempotent of *S*. Now we will obtain the follows implications:

$ks = (e_1 k_1)s \Longrightarrow ks = e_1(k_1 s) \Longrightarrow ks = e_1 y \Longrightarrow ks = e_1(ex) \Longrightarrow ks = (e_1 e)x) \Longrightarrow ks = e'x \Longrightarrow ks \leq x \Longrightarrow x \in (Ks)\omega$

So we have shown also that $(Ks)\omega \supseteq ((K\omega)s)\omega$ and finally we obtain $(Ks)\omega = ((K\omega)s)\omega$.

2. The Maximum Idempotent-Separating Congruence

L. Lallement in [5] has studied the Green's congruences and equivalences in the regulare semigroups the class of which contains the class of the inverse semigroups. Particularly he has studied maximum idempotent-separating congruence in *S* giving this dedfinition:

Definition 3.1: [2], [5], [9], [10], [11] If S is an inverse semigroup and E the set of its idempotents, then the relation $\mu = \{(a,b) \epsilon S^2 : \forall e \in E, a'ea = b'eb\}$ is called maximum idempotent-separating congruence in S.

The above definition was given also by other authors in [2], [9], [10] and [11]. While J. M. Howie in [12] and [2] has given the definition below for the centralizer of a subset H of the semigroup S:

Definition 3.2: [12], [2] If H is a subset of the semigroup S, then it is called centralizer of H in S the set:

$H\zeta = \{s \epsilon S : \forall h \in H, hs = sh\}$

Now we will give the definition of an important class of semigroups that is the class of Clifford semigroups. The Clifford semigroups represent one of the most important types of regular semigroups. Their study was begun in the fundamental paper [13] of A.H. Clifford in 1941. We have this definition:

Definition 3.3: [2],[4],[6], [13] The semigroup S is called a Cloford's semigroup if:

$(\forall x \in S)(\forall e \in E^S)[xe = ex]$

Proposition 3.1: If S is an inverse semigroup with set of idempotents E^S, μ is the maximum idempotent-separating congruence in S, $\rho = \{(a,b)\} \epsilon S^2 : a\mathcal{H}b \wedge ab' \in E\zeta\}$, then $\mu = \rho$ where $\mathcal{H}$ is the known Green's relation.

Proof: First we will show that $\rho \subseteq \mu$. So, let be $(e,f) \epsilon E^S \times E^S$, such that $e\rho f$. Then, from the definition of ρ, we will have $e\mathcal{H}f$ that implies the idempotents e and f belong in the same H class. But, from the corollary 2.6 of the Green's theorem [1], every H class contains no more than one idempotent, so we obtain tha $e = f$ that means ρ is a idempotent-separating congruence in S. Since μ is the maximum idempotent-separating congruence in S, we have $\rho \subseteq \mu$.

Now, we must prove that $\mu \subseteq \rho$. So let be $(a,b) \epsilon \mu$. From the definition of the congruence μ we have $(\forall e \in E^S)[a'ea = b'eb]$. Since aa' is an idempotent i.e. $aa' \in E^S$, we have:

$a'(aa')a = b'(aa')b \Longrightarrow (a'aa')a = b'(aa')b \Longrightarrow a'a = b'(aa')b \Longrightarrow aa'a = a(b'aa'b) \Longrightarrow a = (ab'aa')b \Longrightarrow a = qb \;\; (1)$

where $q = ab'aa'$. In the same way we also show that $b = pa \;\; (2)$, $a = bk \;\; (3)$, $b = al \;\; (4)$. Then, from (1), (2) we obtain $a\mathcal{L}b$ and from (3), (4) we obtain $a\mathcal{R}b$, where $\mathcal{L}$ and $\mathcal{R}$ are the Green's relations in the semigroup S. Now, from the last two relations, $a\mathcal{L}b$, $a\mathcal{R}b$, it follows $a\mathcal{H}b$ (5). It is remain to prove that $ab' \in E\zeta$, i.e. we must to prove that $(\forall e \in E^S)[(ab')e = e(ab')]$. Indeed, suppose that $e \in E^S$ is a random idempotent of S and we obtain:

$(a,b) \epsilon \mu \Longrightarrow a'ea = b'eb \Longrightarrow (a'ea)b' = (b'eb)b' \Longrightarrow a'(eab') = b'bb'e \Longrightarrow a'(eab') = b'e \Longrightarrow aa'(eab') = a(b'e) \Longrightarrow e(aa'a)b' = (ab')e \Longrightarrow e(ab') = (ab')e$

that means $ab' \in E\zeta$ (6). Finally, from (5) and (6) it follows that $(a,b) \epsilon \rho$ and we conclude that $\mu \subseteq \rho$, so we have proved that $\mu = \rho$.

The next proposition below is immediate consequence of the theorem 2.6 in [12] in which J. M. Howie has shown that if S is an inverse semigroup, μ is the maximum idempotent-separating congruence on S and E the set of idempotents of S, then $S/\mu \cong E$ if and only if E is central in S that means $E\zeta = S$ or $(\forall x \in S)(\forall e \in E^S)(xe = ex)$ from which it follows that S is a Clifford's semigroup. In this paper we have represent another proof that we are showing below:

Proposition 3.2: If S is an inverse semigroup then S is a Cliford's semigroup if and only if S/μ is isomorphic to the set of idempotents of S where μ is the maximum idempotent-separating congruence in S.

Proof: First we suppose that S is a Cliford's semigroup. This means that:

$(\forall x \in S)(\forall e \in E^S)(xe = ex)$

Now let be $\bar{a}$ an equivalence class according to the relation μ. So we will have:

$\forall e \in E^S, a'ea = (a'aa')ea = (a'a)(a'e)a = (a'a)(ea')a = (a'a)e(a'a)$

That's mean $a \, \mu \, a'a$. So in every class $\bar{a}$ of equivalence according to the relation μ there is at least an idempotent $a'a$. Since μ is a maximum idempotent-separating congruence in S, it follows that every μ-class $\bar{a}$ has one and only one idempotent. Now, let be I a set of indexes of the set E^S, so $E^S = \{\bar{e_\iota} : i \in I\}$ and let be $\phi: S/\mu \longrightarrow E^S$, where $\forall \bar{e_\iota} \in S/\mu$, $\phi(\bar{e_\iota}) = e_i$. From this definition it is evident that ϕ is a bijection. Furthemore we see that:

$\phi(\bar{e_\iota} \cdot \bar{e_J}) = \phi(\overline{e_\iota e_J}) = e_i e_j = \phi(\bar{e_\iota})\,\phi(\bar{e_J})$

that is mean the map ϕ constructed above is an isomorphism, i.e. $S/\mu \cong E^S$.

Now, let suppose that $S/\mu \cong E^S$ and let prove that S is Cliford's semigroup. Indeed, if $x \in S$ and $e_i \in E^S$ are two random elements, where E^S is the set of idempotents of S, it is enough to prove that $xe_i = e_i x$. Since $S/\mu \cong E^S$ there exists an isomorphism $\phi: S/\mu \longrightarrow E^S$ and each μ −class $\bar{x}$ is idempotent in S/μ and contains an unique idempotent e_j from E^S, that is mean $\bar{x} = \bar{e_J}$ and $\bar{x} = \Phi^{-1}(e_j)$. Now we can write:

$\phi(\overline{xe_\iota}) = \phi(\bar{x} \cdot \bar{e_\iota}) = \phi(\bar{e_J} \cdot \bar{e_\iota}) = \phi(\overline{e_J e_\iota}) = \phi(\overline{e_\iota e_J}) = \phi(\bar{e_\iota} \cdot \bar{e_J}) = \phi(\bar{e_\iota} \cdot \bar{x}) = \phi(\overline{e_\iota x})$

so $\phi(\overline{xe_\iota}) = \phi(\overline{e_\iota x})$, moreover ϕ is an isomorphism that means in injective, hence we obtain:

$\overline{xe_\iota} = \overline{e_\iota x} \Leftrightarrow xe_i \, \mu \, e_i x \Longrightarrow [(xe_i)' \, e_j(xe_i) = (e_i\, x)' \, e_j(e_i x) = (x'e_i) \, e_j(e_i x)$

where e_j is a random idempotent from E^S. So, for the idempotent xx' we can write:

$(e_i x')(xx')(xe_i) = (x'e_i)(xx')(e_i x) \Longrightarrow e_i x' x e_i = x' e_i x \Longrightarrow (x'x)e_i = x'e_i x \Longrightarrow (xx'x)e_i = (xx')e_i x \Longrightarrow xe_i = e_i(xx'x) \Longrightarrow xe_i = e_i x$

from which we conclude that S a semigroup of Cliford. In the above equalities we used the fact that the idempotents in inverse semigroups, commute with each other.

3. Conclusions

In this paper we have extended the range of propositions on inverse semigroups by some new modest theorems which help to further explore the structure of this important class of semigroups that are close to the structure of the groups.

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