Analytical Study of Stability and Hopf Bifurcation in Linear Delay Differential Equations

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Received: 13 December 2026 / Revised: 21 January 2026 / Accepted: 5 March 2026 / Published: 25 March 2026

Doi: 10.56345/ijrdv13n105

Abstract

This paper investigates stability and delay-induced stability switching phenomena in linear homogeneous delay differential systems. Time delays introduce memory effects that significantly enrich the qualitative behavior of dynamical systems and may destabilize equilibria by generating oscillations. We first establish delay-independent stability criteria based on matrix measures and norm estimates, providing explicit conditions that guarantee uniform exponential stability for all delay values. Next, a spectral analysis of the associated characteristic equation is performed to identify critical delay values at which purely imaginary eigenvalues arise and Hopf-type stability switching occurs. For the two-dimensional case, closed-form analytical expressions for the critical frequency and the sequence of delay thresholds are derived, allowing an exact characterization of stability boundaries. Numerical simulations illustrate the theoretical results and confirm the predicted transitions from stable behavior to sustained oscillations. The proposed framework offers a simple, general, and computationally efficient methodology for analyzing stability and bifurcation mechanisms in linear delay systems and can be naturally extended to weakly nonlinear models and practical applications.

Keywords: *Delay differential equations, Linear homogeneous systems, Stability analysis, Hopf bifurcation, Stability switching, Spectral methods*

1. Introduction

Delay differential equations constitute an important class of dynamical systems in which the present evolution depends not only on the current state but also on its past history. Such memory effects naturally arise in numerous applications in biology, engineering, economics, epidemiology, and population dynamics. In many real processes, system responses are not instantaneous, and the influence of past states cannot be neglected. Consequently, delay differential equations provide a more realistic and flexible modeling framework than classical ordinary differential equations. From a mathematical viewpoint, the presence of delay renders the phase space infinite dimensional and leads to significantly richer qualitative behavior compared with ordinary differential equations, including stability switches, sustained oscillations, bifurcation phenomena, and even complex or chaotic dynamics. These characteristics make delay systems both theoretically challenging and practically relevant. The mathematical foundations of functional and delay differential equations, including existence, uniqueness, and stability theory, are well established in the classical literature (Hale & Verduyn Lunel, 1993; Kolmanovskii & Myshkis, 1992; Gu et al., 2003). These works provide the functional–analytic framework and many of the analytical tools that remain fundamental in modern delay analysis. The theory of Hopf bifurcation and the onset of periodic solutions was systematically developed by Hassard et al. (1981) and remains a cornerstone of bifurcation theory. Hopf bifurcation plays a crucial role in explaining the transition from steady behavior to oscillatory regimes and has been widely applied to delay systems arising in engineering and biological contexts. Applications of delay differential equations to biological and population models, where delays represent gestation, maturation, or memory effects, are extensively discussed by Kuang (1993) and Erneux (2009). In such models, delays are often responsible for the emergence of oscillations and cyclic behavior that cannot be captured by instantaneous interactions. More recent research has focused on delay–induced stability switching, distributed delays, and higher–dimensional systems. Analytical and numerical techniques for studying stability transitions have been further developed by Li and Wei (2020), while comprehensive theoretical approaches to functional differential equations are presented in Wu (1996). Additional investigations of delay–induced oscillatory behavior in applied and population models can be found in Ruan (2019) and Bélair and Mackey (1983). These studies demonstrate that delays may fundamentally alter system dynamics and generate rich qualitative phenomena. Related studies by the authors on Hopf bifurcation and oscillatory dynamics in predator–prey and logistic-type systems (Guxholli & Shehu, 2025a, 2025b) motivate the present extension of these ideas to delay–dependent linear systems. Besides analytical approaches, numerical and semi–discretization methods for approximating stability regions and computing critical delays have been proposed by Insperger and Stepan (2011). Eigenvalue-based stability analysis and systematic stabilization strategies are developed by Michiels and Niculescu (2007), while a comprehensive functional and nonlinear treatment of delay equations is provided by Diekmann et al. (1995). Collectively, these studies highlight both the theoretical importance and the practical relevance of delay systems. Despite these advances, much of the existing literature concentrates on nonlinear or application-specific models, whereas comparatively less attention has been devoted to a unified and fully analytical treatment of linear homogeneous delay differential systems. Linear models are of particular importance because they arise naturally from linearization of nonlinear systems and often determine the local qualitative behavior near equilibria. Moreover, linear systems allow explicit analytical calculations and provide clearer insight into the mechanisms responsible for stability loss and oscillation onset. Motivated by these considerations, this paper investigates stability and delay–induced stability switching in linear homogeneous delay differential systems. The main objective is to derive explicit, easily verifiable, and computationally efficient stability criteria. First, delay–independent stability conditions are established using matrix measures and norm–based estimates. Next, a spectral analysis of the associated characteristic equation yields general criteria for the occurrence of Hopf-type stability switching. For the two–dimensional case, closed-form analytical expressions for the critical frequency and the sequence of critical delays are obtained. Numerical simulations are presented to illustrate and validate the theoretical results. From a methodological standpoint, linear delay differential systems play a crucial role in bridging general delay theory and practical stability analysis. Since many nonlinear delayed models are locally approximated by their linearizations, analytical results obtained for linear systems often determine the qualitative behavior near equilibrium states. Consequently, the development of explicit stability criteria for linear delay equations provides not only theoretical insight but also practical guidance for studying more complex dynamical models. This analytical perspective clarifies how oscillatory dynamics emerge as system parameters vary and enables precise identification of parameter regions where stability transitions occur. Such analytical descriptions complement numerical investigations by revealing how stability boundaries depend explicitly on system parameters and delay magnitude, thereby supporting a deeper understanding of delay-induced dynamical transitions. Within this analytical framework, delay effects influence stability primarily through their interaction with spectral properties of the characteristic equation, leading to qualitative transitions that cannot be captured by instantaneous models. From both theoretical and applied perspectives, understanding how delays influence stability boundaries is essential for predicting long-term system evolution. While numerical methods remain useful for visualization and approximation, analytical approaches provide structural insight into parameter dependence and stability mechanisms. Consequently, developing explicit analytical criteria remains an important objective in the theory of delay differential equations, especially for linear systems that serve as local approximations of more complex nonlinear dynamics. Furthermore, analytical investigation of delay effects contributes to a deeper theoretical understanding of stability mechanisms and supports the development of reliable predictive tools for complex dynamical systems involving memory-dependent processes.

The main contributions of this paper are summarized as follows. First, we establish delay–independent stability criteria for linear homogeneous delay differential systems using matrix measures, providing explicit and practically applicable sufficient conditions for uniform exponential stability. Second, we derive general analytical formulas for stability switching and Hopf-type bifurcation directly from the characteristic equation, thereby avoiding purely numerical or simulation-based procedures. Third, for the two–dimensional case, we obtain closed-form expressions for the critical frequencies and delay thresholds, allowing an exact computation of stability boundaries. Unlike many existing studies that focus on nonlinear or problem-specific models, the present work provides a unified and fully analytical framework for general linear delay systems, offering both theoretical transparency and computational efficiency.

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2. Mathematical Analysis

In this section, we develop the analytical framework for studying stability and delay-induced stability switching in linear homogeneous delay differential systems.

Consider the system

$x'(t) = Ax(t) + Bx(t-\tau), \ \ t \geq 0.$ (2.1)

where $x(t) \in R^n$. $A, B \in R^{n \times n}$, and $\tau \geq 0$. The trivial equilibrium is $x^* = 0$. The qualitative behavior of the system is completely characterized by the roots of the associated characteristic equation. Seeking exponential solutions of the form $x(t) = e^{\lambda t} \upsilon$ leads to

$\Delta(\lambda, \tau) = det(\lambda I - A - Be^{-\lambda\tau}) = 0, .$ (2.2)

which is a transcendental equation with infinitely many characteristic roots. The equilibrium is asymptotically stable if and only if all roots lie in the open left half-plane.

We first recall a classical delay–independent stability result based on norm estimates and Halanay-type inequalities.

Theorem 2.1 $(see\ Hale\ \&\ Verduyn\ Lunel, 1993, Kolmanovskii\ \&\ Myshkis, 1992)$ Assume that there exist constants $a > 0$ and $b \geq 0$ such that

$\mu(A) \leq -a, \ \ \|B\| \leq b, \ \ b < a.$

where $\mu(\cdot)$ denotes a matrix measure induced by a suitable norm. Then the equilibrium $x^* = 0$ of

$x'(t) = Ax(t) + Bx(t-\tau)$ is uniformly exponentially stable for every $\tau \geq 0$. Moreover, the exponential estimate $\|x(t)\| \leq Me^{-(a-b)t}\|\varphi\|_\infty, \ \ \|\varphi\|_\infty = \max_{\theta\in[-\tau,0]} \|\varphi\|$, holds for some constant $M \geq 1$ independent of t.

Theorem 2.2 $(see\ Hassard\ et\ al., 1981)$

Suppose that there exist $\tau_0 > 0$ and $\omega_0 > 0$ such that:

(i) Non–hyperbolicity condition

$\Delta(i\omega_0, \tau_0) = 0,$

and the root $\lambda_0 = i\omega_0$ is simple.

(ii) Transversality condition

There exists a branch of roots $\lambda(\tau)$ such that $\lambda(\tau_0) = i\omega_0$ satisfying

$\frac{d}{d\tau}\Re(\lambda(\tau))\Big|_{\tau=\tau_0} \neq 0.$

Then, as τ crosses τ_0, the complex conjugate pair $\lambda(\tau), \overline{\lambda(\tau)}$ crosses the imaginary axis transversally and a stability switch of the equilibrium $x^* = 0$ occurs.

To illustrate the theoretical stability switching criterion established above, we consider the two–dimensional system

$A = \begin{pmatrix} -a_1 & 0 \\ 0 & -a_2 \end{pmatrix}, \ \ B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$

In this case the characteristic equation reduces to

$(\lambda + a_1)(\lambda + a_2) + bce^{-2\lambda\tau} = 0,$

which admits an explicit analytical treatment.

Substituting $\lambda = i\omega$ and separating real and imaginary parts yields algebraic relations for ω and τ. This leads to the critical frequency

$\omega_0 = \sqrt{bc - a_1 a_2},$

and to a sequence of critical delays τ_k that determine the stability boundaries. These values explicitly describe the stability switching predicted by the previous theoretical analysis.

2.1 Numerical Illustrations

In the simulations we use $a_1 = 1$, $a_2 = 1.5$, $b = 2$ and $c = 1$, which satisfy $bc > a_1 a_2$ and ensure the occurrence of stability switching. The numerical integration was performed in Maple using the built-in delay differential equation solver. To validate the theoretical predictions, we present representative simulations illustrating the delay–induced stability switching. The aim is to illustrate the delay–induced stability switching predicted by the analytical criteria and to visualize the qualitative behavior of the solutions.

We consider the two–dimensional system

$$x'(t) = Ax(t) + Bx(t-\tau)$$

with

$$A = \begin{pmatrix} -a_1 & 0 \\ 0 & -a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$$

According to the analytical results, this condition ensures the existence of a critical frequency ω_0 and a sequence of critical delays τ_k at which stability switching occurs. The system was simulated using standard delay differential equation solver for different values of the delay parameter τ. Initial functions were chosen continuous on the interval [−τ,0]. For delay values below the first critical threshold τ_0, all trajectories converge exponentially to the equilibrium, confirming the predicted asymptotic stability. As the delay increases past τ_0, the equilibrium loses stability and oscillatory solutions emerge. The phase portraits clearly exhibit the transition from monotone decay to sustained oscillations, which is characteristic of a Hopf-type bifurcation. Figures 1 and 2 illustrate the stable and oscillatory regimes, respectively. The numerical behavior is in full agreement with the theoretical stability boundaries obtained from the characteristic equation. In particular, the observed oscillation frequency matches the analytically computed critical frequency ω_0.

These simulations confirm the accuracy of the theoretical analysis and demonstrate how time delay alone can generate qualitative changes in the dynamics, including stability loss and oscillatory behavior.

Beyond confirming the theoretical predictions, the numerical simulations also provide qualitative insight into the transition mechanism associated with delay-induced instability. As the delay parameter approaches the critical threshold, trajectories exhibit progressively slower convergence, indicating the loss of stability predicted by the analytical conditions. After crossing the critical value, oscillatory behavior emerges with a frequency consistent with the analytically derived critical frequency, illustrating the close connection between spectral properties of the characteristic equation and observable system dynamics. This agreement highlights how analytical and numerical approaches complement each other: analytical results determine stability boundaries, while simulations provide geometric visualization of the dynamical transition. Such combined analysis is particularly useful for interpreting stability switching phenomena in applications where parameter variations may lead to sudden qualitative changes in system behavior.

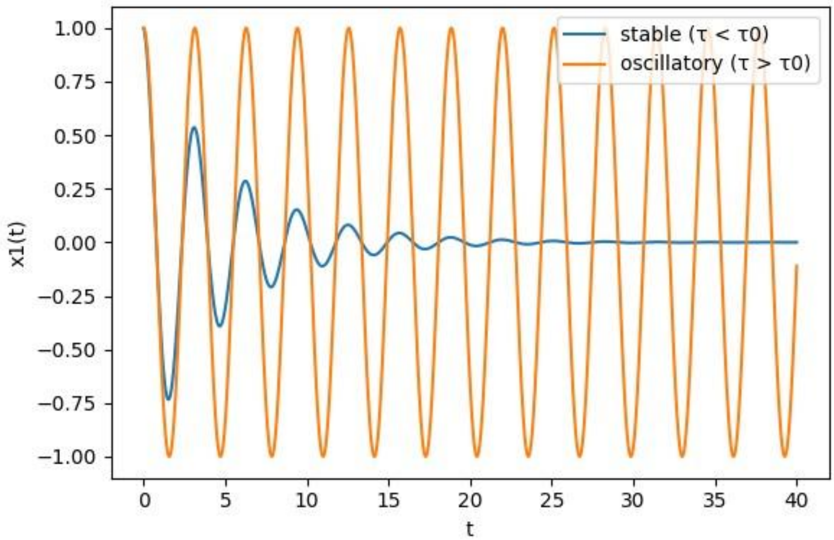

Figure 1. Time series of $x_1(t)$. For $\tau < \tau_0$ the solution converges exponentially to the equilibrium, whereas for $\tau > \tau_0$ sustained oscillations appear.

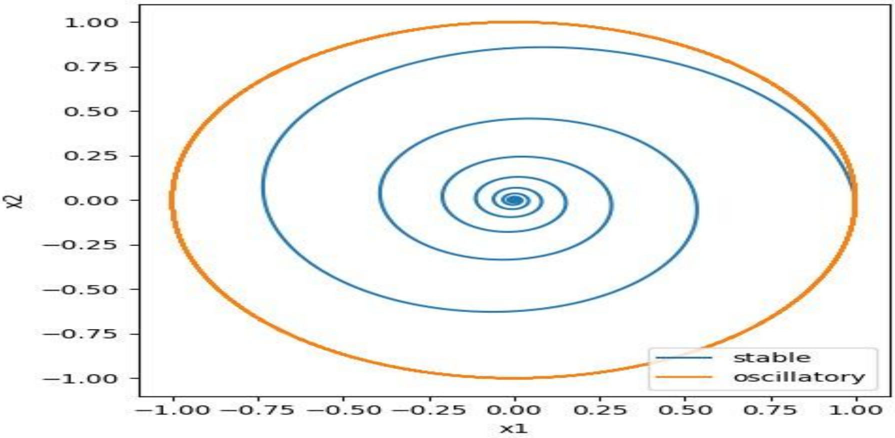

Figure 2. Phase portrait showing spiral convergence in the stable regime and a closed orbit after stability switching.

3. Discussion

The analytical results obtained in this study show that time delays alone can induce substantial qualitative changes in system behavior, even in purely linear models. In particular, the emergence of stability switching and oscillatory dynamics confirms that delays act as intrinsic destabilizing mechanisms capable of transforming an otherwise asymptotically stable equilibrium into a regime characterized by sustained oscillations. This observation highlights the fundamental role of memory effects in many real-world processes, where delayed responses are unavoidable and may significantly influence long-term system dynamics. A central feature of the proposed approach is the use of explicit analytical criteria for stability assessment. While numerical simulations and discretization techniques are widely employed in the literature, they often provide approximate stability boundaries and may obscure the mechanisms responsible for stability loss. In contrast, the delay–independent conditions and closed-form stability switching formulas derived in this work offer direct theoretical insight into the interaction between system parameters and delay values. As a result, stability properties can be analyzed in a transparent and systematic manner, allowing clearer interpretation of the dynamical mechanisms governing the onset of oscillatory behavior. Compared with studies primarily focused on nonlinear or application-specific models, the present work develops a unified analytical framework for general linear homogeneous delay systems. Linear models play a fundamental role because they naturally arise through linearization and frequently determine the local behavior of nonlinear systems near equilibria. Consequently, the theoretical results established here provide a foundation for further investigations of nonlinear dynamics using local bifurcation and perturbation techniques, extending the applicability of the analytical framework beyond purely linear settings. From an applied perspective, the availability of explicit stability boundaries is particularly valuable for parameter selection, design, and control problems in engineering, biological, and population models. Practitioners can estimate admissible delay ranges without relying exclusively on repeated simulations, thereby facilitating model calibration and optimization processes. This advantage becomes especially significant in high-dimensional systems or when delay parameters must be adjusted iteratively during model development. Furthermore, the analytical framework developed in this study is naturally extendable to broader classes of delayed systems. Since nonlinear models are often locally approximated by their linearizations, explicit analytical stability criteria provide essential qualitative information for understanding more complex dynamical phenomena. The results demonstrate that even simple linear delay differential systems may exhibit rich and nontrivial dynamics, emphasizing both the theoretical importance and practical relevance of incorporating delay effects into mathematical modeling. Overall, the proposed methodology establishes a coherent link between analytical theory and practical applications. By combining spectral analysis with delay-independent stability conditions, the approach offers a systematic tool for investigating delayed dynamical systems while preserving interpretability and mathematical transparency of the derived stability conditions. These features make the framework suitable not only for theoretical investigations but also for applied modeling contexts where reliable stability assessment is required.

4. Conclusions

In this paper, we investigated stability and delay–induced stability switching phenomena in linear homogeneous delay differential systems. By combining matrix measure techniques with spectral analysis of the associated characteristic equation, delay–

independent stability conditions ensuring uniform exponential stability of the trivial equilibrium were established. Analytical investigation of purely imaginary characteristic roots allowed the determination of critical delay values at which Hopf-type stability switching occurs, while closed-form expressions obtained for the two–dimensional case provided an explicit characterization of stability boundaries. Numerical simulations confirmed the theoretical predictions and illustrated the transition from stable dynamics to sustained oscillations as the delay parameter increases. The strong agreement between analytical and numerical results demonstrates the consistency and reliability of the proposed framework. The findings show that time delays alone can generate complex dynamical behavior even in linear systems, highlighting the importance of incorporating delay effects into mathematical modeling. The analytical stability criteria developed in this work provide a transparent tool for stability assessment and may assist in parameter selection, model calibration, and control design in applications where delayed responses are inherent. Future research may extend the present framework to systems with multiple or distributed delays, time-varying delays, nonlinear interactions, and higher–dimensional models. These directions may further enhance the understanding of delay-induced dynamical phenomena and support the development of efficient analytical and control strategies for delayed systems.

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